Translation surfaces: saddle connections, triangles and covering constructions

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July 29, 2016

Translation surfaces

A (compact) translation surface is:

- a compact surface with a atlas to $\mathbb{R}^2$ (or $\mathbb{C}$), covering all but finitely many points, whose transition functions are translations, or
- a collection of polygons in the plane with sides glued through translation, or
- a compact Riemann surface with a holomorphic 1-form.

Examples: flat torus, regular octagon with opposite sides glued.
Applications: Rational polygonal billiards, IET.

Strata of translation surfaces

- The sets of translation surfaces with the same topological type are called strata. They are complex affine manifolds where the charts are period coordinates. They are never compact and in general not connected.
- The group $SL(2,\mathbb{R})$ acts on a stratum by post-composing with the coordinate charts. This action renormalize the geodesic flow on surface. All $SL(2,\mathbb{R})$ orbits are known to be complex affine submanifolds (Eskin-Mirzakhani-Mohammadi).
- The horocycle orbit closures are also important. However, these are not always affine submanifolds.

Lattice Surfaces

- Surfaces with closed $SL(2,\mathbb{R})$ orbit are called lattice surfaces. The simplest of these is the flat torus. Branched covers of the flat torus that are lattice surfaces are called square-tiled.
- Non-square tiled lattice surfaces: regular polygon (the first, discovered by Veech), genus 2 lattice surfaces, prym eigenform in $\mathcal{H}(4)$ or $\mathcal{H}(6)$, Bouw-Möller family etc.

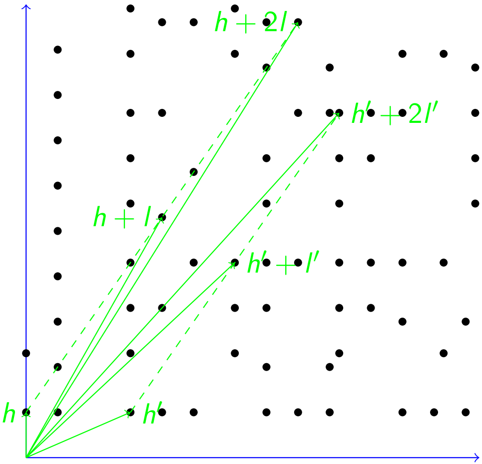
Veech Dichotomy

- Given a direction on a translation surface, one can define the geodesic flow on the surface in that direction for almost all points.
- Veech Dichotomy: The geodesic flow on any lattice surface must be either uniquely ergodic or completely periodic (i.e. the surface is decomposed into cylinders in that direction).
- When it is completely periodic, it is easy to show that furthermore the moduli of the cylinders must be commensurable.

Holonomies of saddle connections

- Example: Flat torus with a marked point.
- Example: a lattice surface in $\mathcal{H}(2)$ corresponding to the quadratic order of discriminant 13. (p.6)
- Quadratic growth rate.
- The holonomies of saddle connections on a non-square-tiled lattice surfaces can not be *uniformly discrete*, i.e. the distance between two distinct points are not bounded below.
- For square-tiled case, it is a classical result that the set is not *relatively dense*.

Holonomies of saddle connections

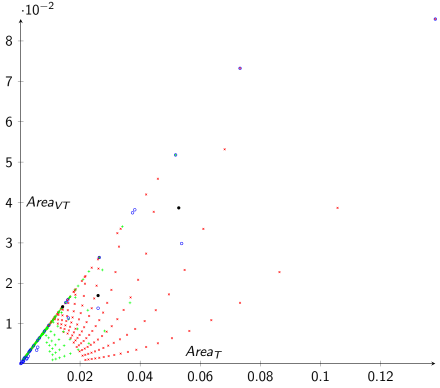


Minimal area of triangles and virtual triangles

- Smillie-Weiss, Vorobets showed that the area of triangles and "virtual triangles" on a lattice surface must be bounded away from 0 and the converse is also true.
- Smillie and Weiss described an algorithm that can enumerate all lattice surfaces according to the area of smallest virtual triangle, represented through Thurston-Veech construction.

Minimal area of triangles and virtual triangles

We did computations on this area for some known lattice surfaces, using known enumeration of cylinder decompositions on these surfaces up to affine action.



Thurston-Veech construction

- If a surface is completely periodic in two different directions, and all cylinder moduli in each direction are commensurable, then it can be uniquely determined up to affine transformation by the intersection pattern of the cylinders and the ratio of the moduli. This is called a *Thurston-Veech construction*.
- The intersection pattern can be represented by a ribbon graph, called *Thurston-Veech diagram*.
- Thurston used it to construct pseudo-Anosov diffeomorphisms.

Algorithm for enumerating lattice surfaces according to smallest virtual triangle

- Step I: find all possible cylinder intersection pattern with less than a certain number of cylinder intersections (i.e. rectangles).
- Step II: find all possible ratios of moduli.
- Step III: check for the area of the smallest virtual triangle.

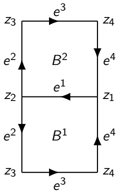
We improved the bound in Step II and did part of Step III by computer while the remaining by hand, and found all non-arithmetic lattice surfaces with area of smallest virtual triangle at least .05 of the total area. There are 5 of them up to affine action.

Branched Abelian cover of the flat pillowcase

An important class of example of square-tiled lattice surfaces are the Abelian branched covers of the pillowcase (which is a *half translation surface* i.e. the transition function is a translation or rotation of π).

- They are used in Bouw-Möller's construction of a two-parameter infinite family of lattice surfaces, which includes the Veech & Ward families.
- Their Lyaponov exponents have been calculated.
- They can be used to construct interesting examples of horocycle orbit closures. (See Lucien's thesis)
- They are related to the classical problem of hypergeometric differential equations (See Deligne-Mostow).

Branched Abelian cover of the flat pillowcase



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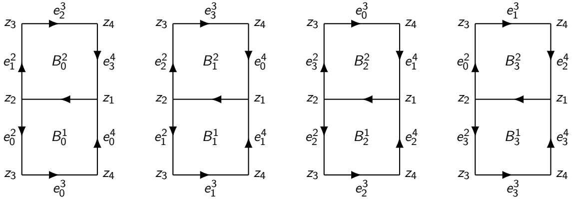


Branched Abelian cover of the flat pillowcase

Results:

1. $$H^1(M,\Sigma;\mathbb{C}) = \bigoplus_{\rho\in\Delta} H^1(\rho)$$

   this decomposition is preserved by the action of the affine group **Aff**, while the factors may be permuted.
2. The dimension of $H^1(\rho)$ is 3 if ρ is the trivial representation and 2 otherwise.
3. The projection to absolution cohomology restricting to each $H^1(\rho)$ is either trivial or does not split.
4. For ρ such that the map in 3. is not 0, a finite index subgroup of the affine group acts on $H^1(\rho)$ as a (Euclidean (when the map in 3. does not split), hyperbolic or spherical) triangle group.

The case for $H^1(M)$ was done by Wright.
In the case where the action is a spherical triangle group, we can decide if it is discrete by ADE classification.

Bouw-Möller surfaces

Bouw-Möller construction. Hooper's grid graph.
If a lattice surface arises from deforming an abelian branched cover of flat pillowcase M using a relative cohomology class α, its affine diffeomorphism group is with the affine diffeomorphism group of M, and the deformation also preserves the Thurston-Veech structure in the two diagonal directions, then this surface be one of the followings cases:

1. a branched cover of the regular $2n$-gon branched at the cone point.
2. a branched cover of a Bouw-Möller surface.
3. a branched cover of the regular n-gon branched at the mid-point of edges.

Bouw-Möller surfaces

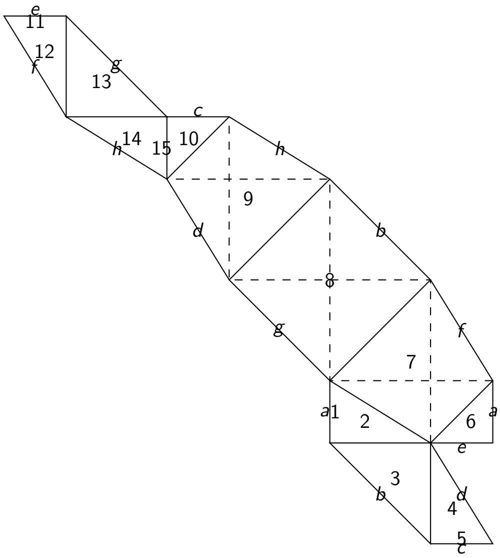


Figure: The $(5, 3)$ Bouw-Möller surface.

"Flipping"

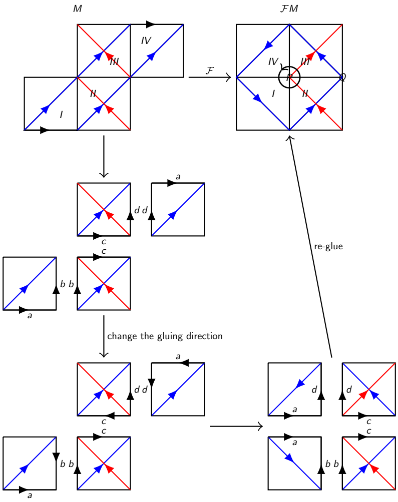
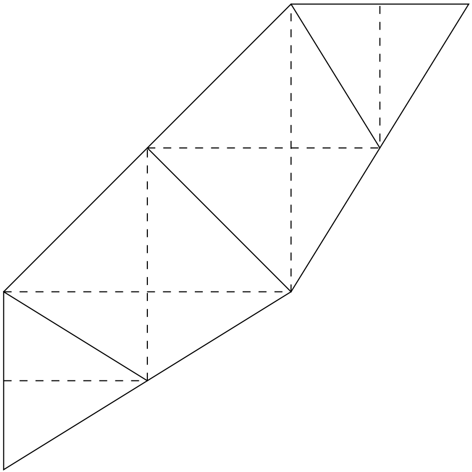


Figure: Constructing Thurston-Veech diagram by flipping.

The case 3.



Thank you very much!