Research Statement

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1 Introduction

My research has so far been mostly focused on the dynamics and geometry of translation surfaces, low dimensional topology, and tropical geometry. Below is a list of some of my ongoing projects. More background and details will be provided in the next section.

- **A Borel-Serre type compactification of the strata.** I am working with my PhD advisor Prof. John Smillie on developing a way to compactify the strata of finite translation surfaces, and exploring its possible application on the dynamics of unipotent flow.
- **Upper bound on the translation length on curve complexes.** I am working with Hyungryul Baik and Hyunshik Shin on finding good upper- and lower- bounds on the translation lengths of pseudo-Anosov maps on a fibered cone.
- **Entropy of post-critically finite generalized β-maps.** I am working with Harry Bray, Diana Davis, and Kathryn Lindsey on generalizing results on the set of Galois conjugate of slopes of all PCF unimodular maps, all β-maps as well as all generalized β-maps, to the set of Galois conjugates of PCF generalized β-maps that admits various constrains. This is a project supported by MRC funding.
- **Foliations on 3-manifolds from left orderability** Harry Baik constructed a singular foliation for 3-manifolds with 1-vertex triangulation and left orderable π_1. With Harry Baik and Sabastian Hensel we are trying to understand the possible implictions of this construction.

And below are some of my previous projects that are either finished or in the process of write-up. More details about them will be provided in section 3.

- **A Kazhdan's theorem for graphs, manifolds and simplices.** The Kazhdan's theorem [M] stated that the hyperbolic metric on a compact Riemann surface can be obtained as a limit of the canonical metric on its finite covers. Motivated by the Birkovich space and Mumford curves, Farbod Shokrieh and I proved a result on finite metric graphs that is analogous to Kazhdan's theorem, and generalized our method to simplicial complexes and Riemannian manifolds of any dimensions. We are in the process of writing up our results.

- **Constructing pseudo-Anosov maps and Teichmüller polynomials** Harry Baik, Ahmad Rafiqi and I [BRW] gave a sufficient condition for constructing pseudo-Anosov maps by "thickening" a post-critically finite continuous interval map, and an efficient algorithm for checking this condition, and when the condition is satisfied, calculate their Teichmuller polynomials. We also found connection with end-periodic maps and depth-1 foliations.

- **Abelian covers of the flat pillowcase.** I calculated the affine group action on the Abelian cover of flat pollowcases, and, as application, found a more geometric interpretation of the Bouw-Möller construction, and together with Lucien Clavier, found a new example of a horocycle orbit closure in strata, which has self-intersection.

- **Topology of wild singularities.** Together with Anja Randecker and Lucien Clavier, I studied the topology of rotational components of many wild singularities of the infinite translation surface, and found many interesting examples.

- **Area of smallest triangles and virtual triangles.** I calculated the smallest area of triangle and virtual triangle on many lattice surfaces, implemented the algorithm in [SW] and used it to find all lattice surfaces with area of smallest virtual triangle no less tham 0.05.

- **Non-uniform discreteness of the holonomy vectors.** I showed that the holonomy vectors of non-arithmetic lattice surfaces can not be uniformly discrete.

- **Dilatation of psuodo-Anosov maps and bi-Perron numbers.** Together with Harry Baik and Ahmad Rafiqi, we showed that asymptotically some proportion of the bi-Perron numbers of a given degree can not be pseudo-Anosov dilatation of a surface with low genus.

2 Ongoing Projects

2.1 A Borel-Serre type compactification of the strata

A *translation surface* is a surface with a translation structure, which is a local chart that covers the surface except for a discrete set Σ, for which the change of coordinate maps are translations. For example, holomorphic differentials on compact Riemann surfaces make them translation surfaces. The translation surfaces constructed in this way are called *finite translation surfaces*, while the other translation surfaces are called *infinite translation surfaces*. Translation surfaces arise naturally in the study of certain dynamical systems, Teichmüller theory, and the theory of hyperbolic 3-manifolds.

The set of holomorphic differentials with zeros of orders $k_1, \ldots, k_j$ up to permutation is called the stratum $\mathcal{H}(k_1, \ldots, k_j)$. The stratum is a complex affine manifold through the period coordinates. The group $GL(2, \mathbb{R})$ acts on the stratum by post-composition with the charts of the translation surfaces. This action preserves the measure induced by the periodic coordinates, and is ergodic with respect to an invariant measure defined through period coordinate. Since [EMM], there have been many major breakthroughs on the study of orbit closures of this group action by Wright, Filip, and others.

An approach to study the strata of translation surfaces is through its boundary, which requires a "good" compactification. Many useful compactification of the strata have been constructed e.g. [BCGGM]. My PhD advisor John Smillie developed a way to compactify the stratum similar to the Borel-Serre compactification, which embedded the strata into a compact real orbifold with corners. We are currently working on understanding the properties of this compactification. For example, we described the strata of meromorphic differentials that contain differentials with zero residue at all the cone points, and showed that:

Theorem 1
Let $\mathcal{H} = \mathcal{H}(n_1, \ldots, n_r, -p_1, \ldots, -p_s)$ with genus $g = 1 + \frac{1}{2}(\sum_i n_i - \sum_j p_j)$.

(i) When $g > 0$, $\mathcal{H}$ contains elements with zero residue at all poles iff $p_j > 1$ for all j.

(ii) When $g = 0$, $\mathcal{H}$ contains elements with zero residue at all poles iff $p_j > 1$ for all j, and $n_i \leq \sum_j p_j - s - 1$ for all i.

We also built affine manifold models of this compactification in a few small strata. This compactification can also be used to give an alternative proof of Theorem 6 in Section 3.1. Furthermore, this construction can be generalized to the compactification of some affine submanifolds of $\mathcal{H}$. However, the action of $GL(2, \mathbb{R})$ on $\mathcal{H}$ do not extend onto it and we are still working on describing the closure of a $GL(2, \mathbb{R})$-orbit closure in our compatification as is done by Mirzakhani-Wright in their bordification. Furthermore, we are also working on using this bordification, and the local product structure of the boundary pieces, to understand the relationship between fundamental groups of the different strata.

2.2 Upper bound on the translation length on curve complexes.

A pseudo-Anosov map on a closed surfaces with Euler characteristic χ induces a map on its curve complex. The asymptotic translation length of this map on the curve complex is shown to be positive by [MM], to be a rational number with bounded denominator by [Bowd], and to be bounded below up to a constant by $\frac{1}{\chi^2}$ by [GT]. For pseudo-Anosov maps corresponding to the primitive integer points in a fibered cone of a hyperbolic 3-manifold, [KS] found an arithmetic sequence in this set whose asymptotic translation length in the corresponding curve complexes are bounded above by $1/\chi^2$ up to a multiplicative constant, where χ is the Euler characteristic of the surface.

With Hyungryul Baik and Hyunshik Shik, we are able to generalize the argument in [KS] further to any closed subcone of any 2-dimensional section of the fibered cone, and prove the following:

Theorem 2
Let F be a fibered cone in $H^1(M)$ for some 3-manifold M, for any 2-dimensional plane H passing through the origin, defined by equations with rational coefficients, and such that $H \cap F \neq \emptyset$, let $\mathcal{C} \subsetneq F_H = F \cap H$ be any closed subcone, there is some constant $C > 0$ depending on M and $\mathcal{C}$ such that for any primitive integer chomology class $\alpha \in \mathcal{C}$, the asymptotic translation length of the monodromy of the fiberation corresponding to α is bounded above by $\frac{C}{\chi^2}$ where χ is the Euler characteristic of the fiber.

Currently we are working on extending our result in the following two directions:

(i) Can there be a similar bound on any closed subcone of the fibered cone, in the case when this fibered cone has dimension greater than 2?

(ii) It follows from [GHKL] and [H] that there are arithmetic sequences in a direction parallel to the boundary of the fibered cone so that the asymptotic translation length in curve complex decay on the order of $\frac{O(1}{\chi)}$. Is it true for any such arithmetic sequences?

2.3 Entropy of post-critically finite generalized β-maps

A generalized β-map is a piecewise linear map from the unit interval to itself with slope $\pm\beta$, which sends $(\frac{j-1}{\beta}, \frac{j}{\beta})$ to the whole interval linearly for $j = 1, \dots, \lfloor\beta\rfloor$, and send $(\frac{\lfloor\beta\rfloor}{\beta}, 1)$ linearly to $(0, \beta - \lfloor\beta\rfloor)$ if it has positive slope and $(1-\beta+\lfloor\beta\rfloor, 1)$ if it has negative slope. When all the slopes are positive it is called a β-map, when all slopes are negative a negative β-map. The unimodal interval map is a special case of the generalized *beta*-map. If the forward orbit of 1 under this map become eventually periodic, it is called post-critically finite (PCF).

The closure of the Galois conjugates of the slopes of PCF unimodal maps has been studied in [MT, T, Ti, CKW] etc. The closure of the Galois conjugates of the slopes of all PCF β-maps was described in [So] and an analogous description for the case of all PCF generalized β maps is given in [Thom]. The unimodal case is very different from the latter two cases, for example, the closure in the unimodal case has at least one hole as shown in [CKW] while in the other two cases the closure consists of a region bounded by a simple closed curve, so it would be interesting to describe some "intermediate" situations and to see how one evolve into another. Harry Bray, Diana Davis, Katheryn Lindsay and I have been working on generalizing results from these three sets to the set of

PCF generalized β-maps that admits additional constraints (e.g. a bound on the size of β). As a first step, we showed that the condition on the itinaries for the symbolic dynamics of a generalized β map in [G], which was found to be not sufficient in [St], is nevertheless sufficient for our purpose of calculating Galois conjugates. Furthermore, we found a connection between how fast the interior of the closure is being "filled in" as one increases the bound on the period and the discrete subrings of $\mathbb{C}$.

2.4 Foliations on 3-manifolds from left orderability

Let M be a cellular complex with a one-vertex triangulation, if its fundamental group $\pi_1(M)$ admits a left invariant ordering, one can obtain a singular foliation on M with the following procedure: first pick an embedding of $\pi_1(M) \to \mathbb{R}$ which preserves this order, then, define a map from the universal cover $\tilde{M}$ to $\mathbb{R}$ by identifying the preimages of the vertex with elements of $\pi_1(M)$ and send them to $\mathbb{R}$ by i, and then extend the map from vertices to higher dimensional simplices piecewise linearly, and let the leaves of the foliation be the level curves. It can be verified that this foliation on $\tilde{M}$ is the pullback of a foliation on M. In the case when M is a manifold, this is a foliation with one single singular point.

Hyungryul Baik, Sebastian Hensel and I constructed many examples of left invariant orderings on π_1 and the corresponding foliations in the case when M is a surface or a three manifold, and we are trying to prove more general statements regarding this construction.

3 Previous Projects

3.1 A Kazhdan's theorem for graphs, manifolds and simplices.

Kazhdan's theorem for compact Riemann surface theory says that the hyperbolic metric from uniformization can be obtained alternatively via the Jacobians of finite covers, as the limit of the canonical metric inherited from increasingly larger finite Galois over. c.f [M]. The canonical metric is related to the distribution of Weierstrass points as well as to Arakelov theory. Farbod Shokrieh and I found a way to obtain a similar result to Zhang's canonical measures [Z, BF] on finite metric graph. We showed the following:

Theorem 3
Let $\phi\colon \Gamma' \to \Gamma$ be a infinite Galois covering of compact metric graph Γ. Let $\phi_n\colon \Gamma_n \to \Gamma$ (for $n \geq 1$) be an ascending sequence of finite Galois covers converging to Γ', in the sense that the equality

$$\bigcap_{n\geq 1} \pi_1(\Gamma_n) = \pi_1(\Gamma') \tag{1}$$

holds in $\pi_1(\Gamma)$*. Let* μ'_{can} *be the canonical measure on* Γ'*,* μ^i_{can} *be the canonical metric on* Γ_i*. Let* μ_ϕ *and* μ_{ϕ_i} *be measures on* Γ *whose pull-backs are* μ'_{can} *and* μ^i_{can}*, respectively. Then*

$$\lim_{n\to\infty} \mu_{\phi_n} = \mu_\phi \ .$$

A key lemma for our theorem is the "Gauss-Bonnet's theorem" for canonical measures on infinite metric graph:

Theorem 4
Let $\phi\colon \Gamma' \to \Gamma$ *be a infinite Galois covering of compact metric graph* Γ*. Let* μ'_{can} *denote the canonical measure on* Γ'*, and let* μ_ϕ *be the measure on* Γ *whose pull-back is* μ'_{can}*. Then*

$$\mu_\phi(\Gamma) = -\chi(\Gamma) \ .$$

which can be proved through either harmonic analysis on infinite metric graphs or von-Neumann algebra, and together with Harry Baik, we managed to prove analogous theorems for any metrized simplicial complexes or Riemannian manifolds.

3.2 Constructing pseudo-Anosov maps and Teichmüller polynomials

The works of de Carvalho and T. Hall [CH] described a way of constructing a pseudo-Anosov map by "thickening" a post-critically finite continuous interval map into a piecewise linear map in 2 dimension, then gluing the boundary of its ω-limit set to make it a continuous surface map. A similar construction was also described by Thurston [T]. However, in general, the translation surface one gets in this way is not a finite translation surface. Harry Baik, Ahmad Rafiqi and I gave a sufficient condition for this construction to give us a finite translation surface in [BRW]. We also implemented an algorithm to check our condition efficiently, and obtained many examples.

Furthermore, we found and implemented an efficient algorithm for calculating their Teichmüller polynomials [M2], by relating the post-critically finite interval map and the invariant train-track and then utilize Fox calculus.

Following the suggestion of Danny Calegari, we built many examples of pseudo-Anosov maps on finite surfaces from those interval maps that give us infinite translation surfaces, by going through end-periodic maps. We start by "blowing up" the ends of the finite translation surface to reverse the Handel-Miller construction, so that the affine map lifts to an end-periodic map of a larger surface. Then, we build a 3-manifold with toric boundaries with a depth 1 foliation as in [F] and perturb it to get a sequence of pseudo-Anosov maps. In particular, we showed that the sequence described by Bowman in [Bowm] is an example of this construction.

3.3 Abelian cover of the flat pillowcase

A finite translation surface whose $GL(2,\mathbb{R})$ orbit is closed is called a *lattice surface*, and the discrete subgroup of $SL(2,\mathbb{R})$ formed by the derivatives of its affine diffeomorphisms is called its *Veech group*. One of the first classes of examples of lattice surfaces is the square-tiled surfaces, which are branched covers of the flat torus with one single branch point. The Veech groups of square-tiled surfaces are finite index subgroups of $SL(2, \mathbb{Z})$, and the algorithm to calculate them was described in [S].

A class of square-tiled surfaces that has been extensively studied is the abelian branched covers of the flat pillowcase. For example, [Wr1] studied the Veech group action on their cohomology; [MY] calculated the full affine diffeomorphism group action on the relative homology of two well-known translation surfaces in this class, the Wollmilchsau and the Ornithorynque; and [BM] used them to construct and study a class of non-square-tiled lattice surfaces which was further described in [Wr2]. In order to answer a question raised by Smillie and Weiss in their study of horocycle orbit closure, I calculated the affine diffeomorphism group action on the relative cohomology of an abelian branched cover of the flat pillowcase, and showed the following:

Theorem 5 ([Wu1])

(i) *Let M be a branched cover of the pillowcase with abelian deck group G. Let $\Sigma \subset M$ be the preimage of the four cone points of the pillowcase. Let Δ be the set of irreducible representations of a finite abelian group G. The deck transformations give an action of G on $H^1(M, \Sigma; \mathbb{C})$. Let $H^1(\rho)$ be the sum of all irreducible sub-representations of $H^1(M, \Sigma; \mathbb{C})$ that are isomorphic to ρ. Then, we have decomposition $H^1(M, \Sigma; \mathbb{C}) = \bigoplus_{\rho \in \Delta} H^1(\rho)$, which is preserved by the action of the affine group* **Aff***, though the factors may be permuted.*

(ii) *Let g_i, $i = 1, 2, 3, 4$ be the elements in deck group G that correspond to the counterclockwise loops around the four cone points of the pillowcase. Then, the projection $r : H^1(M, \Sigma; \mathbb{C}) \to H^1(M; \mathbb{C})$ splits as an* **Aff***-module if and only if deck group G does not have a representation ρ such that only one of the four $\rho(g_i)$ is 1.*

(iii) *In the case when* **Aff** *action on $H^1(\rho)$ preserves a positive definite Hermitian norm, this action is discrete if and only if $\rho(g_1) = \rho(g_2)$, $\rho(g_3) = \rho(g_4)$ up to permutation, or if it is one of the other sporadic 28 cases.*

As a consequence, I gave a positive answer to the question by Smillie and Weiss:

Theorem 6 ([Wu1])
There is a square-tiled surface M with the set of cone points Σ constructed as a normal abelian branched cover of the flat pillowcase, such that:

(i) *There is a direct sum decomposition $H^1(M, \Sigma; \mathbb{R}) = N \oplus H$ preserved by the action of the group of orientation preserving affine diffeomorphisms.*

(ii) *There is a positive definite norm on N invariant under the affine diffeomorphism group action.*

(iii) *The affine diffeomorphism group action on N does not factor through a discrete group.*

Following [BM, Wr2], I used the constructions in [Wu1] to study the Bouw-Möller surfaces, and found an alternative characterization of them as well as a more explicit construction of Hooper's "grid graph". In [Wu2], I defined a concept of "deformation" and showed that the lattice surfaces arise from "deforming" an abelian branched cover of flat pillowcase which also satisfy two other conditions must be branched covers of the Bouw-Möller surfaces.

As another application, Lucien Clavier and I did some calculation on horocycle orbit closures arise from "pushing" the $SL(2,\mathbb{R})$ orbit of some square-tiled surfaces. Some of the results are summarized below:

Theorem 7
There is a horocycle orbit closure in $\mathcal{H}(1,1,1,1)$, which is an immersed affine submanifold with boundary, such that there are infinitely many boundary components within finite distance, and that it intersects itself transversely.

This result is proved by using the relationship of interval exchange maps and translation surfaces. It can also be understood from the perspective of Borel-Serre type compactification of the strata c.f. Section 1.1.

3.4 Topology of wild singularities

An isolated singularity of a translation surface must be one of the 3 cases: a cone point of finite cone angle, where locally it is a $\mathbb{Z}/n$-branched cover of an open disc branched at the center; a cone point of infinite cone angle, where locally it is a $\mathbb{Z}$-branched cover of an open disc branched at the center; or neither of the above, in which case it is called a *wild singularity*. To study the topology of a neighborhood of a wild singularity, [BV] introduced the concept of linear approaches, which are divided into rotational components. [BV] also defined a topology on the set of linear approaches. Anja Randecker, Lucien Clavier and I investigated the quotient topology on the set of rotational components at one wild singularity, and its relationship with the topology of the set of linear approaches, and found many interesting examples. In particular, we showed the following:

Theorem 8 ([CRW])

(i) *Any topological space of finitely many points is homeomorphic to the topology of rotational components at some wild singularity.*

(ii) *There are uncountably many wild singularities with the same topology on the space of rotational components but different topologies on the space of linear approaches.*

There are still many remaining questions regarding this project. In particular, we hope to have a criteria for which kind of topologies the set of rotational component can or can not have, and we hope to use these concepts to develop a concept of strata for infinite translation surfaces. Furthermore, it seems to us that there may be some connection between the concept of rotational component and the "blow up" construction described in Section 1.2.

3.5 Area of smallest triangles and virtual triangles

Vorobets, Smillie and Weiss [SW] showed that a surface is a lattice surface if and only if there is a lower bound on the areas of embedded triangles such that all vertices are cone points, or, if and only if there is a lower bound on the length of the cross product of two non-parallel saddle connections. The former is called the area of the smallest triangle while one half of the latter is called the area of the smallest virtual triangle. In [Wu3], I calculated the area of the smallest triangle and virtual triangle of lattice surfaces in $\mathbb{H}(2)$, in the Prym loci of $\mathcal{H}(4)$, as well as the Bouw-Möller family. Furthermore, I optimized the algorithm described in [SW] and used it to show that all lattice surfaces with the area of smallest virtual triangle greater than $1/20$ are already known.

3.6 Non-uniform discreteness of holonomy vectors

There is a question posed by Barak Weiss on if the set of holonomy vectors of translation surfaces is always uniformly discrete. I gave a negative answer by showing:

Theorem 9 ([Wu4])
The holonomy vectors of a non-square-tiled Veech surface can not be uniformly discrete.

3.7 Dilatation of psuodo-Anosov maps and bi-Perron numbers

With Harry Baik and Ahmad Rafiqi, we showed that:

Theorem 10 ([BRW2])
Let $D_n(R)$ be the set of dilatations of pseudo-Anosov maps of a surface of genus n which are no larger than n, $B_n(R)$ be the set of bi-Perron numbers no larger than R, then $\lim_{R\to\infty}|\frac{D_n(R)\cap B_n(R)}{B_n(R)}| = 0$ when $n > 9$.

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