2.1 Instantaneous rate of change, or, an informal introduction to derivatives

Let a, b be two different values in the domain of f. The *average rate of change* of f between a and b is $\frac{f(b)-f(a)}{b-a}$.

Geometrically, a *secant line* of the graph of a function f is a line that passes through two different points on the graph $(a, f(a))$ and $(b, f(b))$. The slope of the secant line is the average rate of change of f between a and b.

If, as b approaches a, the average rate of change between a and b approach a number d, then d is called the *instantaneous rate of change*, or the *derivative* of f at a.

Geometrically, when we fix one of the two points, say fixing a, and let b approaches a, if the secant line converges to a line l_a, then l_a is called the *tangent line* of the graph of f at point $(a, f(a))$. The slope of the tangent line is the instantaneous rate of change.

Example: $f(x) = x^3$

Example: $g(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$

2.2 Limit Laws

$\lim_{x\to c} f(x)$ only depends on the value of f on a open interval containing c excluding c.

$\lim_{x\to c} C = C$, where C is a constant.

$\lim_{x\to c} x = c$

When $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, $\lim_{x\to c}(f+g)(x) = \lim_{x\to c} f(x)+\lim_{x\to c} g(x)$, $\lim_{x\to c}(f-g)(x) = \lim_{x\to c} f(x)-\lim_{x\to c} g(x)$ and $\lim_{x\to c}(fg)(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$. As a consequence, if P is a polynomial, $\lim_{x\to c} P(x) = P(c)$.

When $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, and the latter is non-zero, $\lim_{x\to c}(f/g)(x) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$. As a consequence, if P, Q are both polynomial and $Q(c) \neq 0$, $\lim_{x\to c}(P/Q)(x) = P(c)/Q(c)$.

Example: $\lim_{x\to 1} \frac{x^2-1}{x^3-1} = \lim_{x\to 1} \frac{x+1}{x^2+x+1} = 2/3$.

When $\lim_{x\to c} f(x)$ exists, $\lim_{x\to c} f^n(x) = (\lim_{x\to c} f(x))^n$, $\lim_{x\to c} f^{\frac{1}{2n+1}}(x) = (\lim_{x\to c} f(x))^{\frac{1}{2n+1}}$.

When $\lim_{x\to c} f(x) > 0$, $\lim_{x\to c} f^{\frac{1}{2n}}(x) = (\lim_{x\to c} f(x))^{\frac{1}{2n}}$.

If $f \leq g$, $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$. (note: we can't replace $\leq$ with $<$ here.)

Sandwich theorem: if $f \leq g \leq F$, $\lim_{x\to c} f(x) = \lim_{x\to c} F(x) = L$, then $\lim_{x\to c} g(x) = L$.

Example: $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Classification of the cases when the limit does not exist:

1: Unbounded

2: Bounded but "jumps" (i.e. both left & right hand limit exist but they are different.)

3: All remaining cases. We call them "oscillate too much".

2.3 The definition of limit

Suppose f is defined on an open neighborhood of a except possibly at a. By "$\lim_{x\to a} f(x) = L$", we mean $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x$, $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$. ($\forall$: for all $\exists$: there exists)

Example: $\lim_{x\to c} x = c$ [Let $\delta = \epsilon$.]

As a consequence, by "$\lim_{x\to a} f(x)$ does not exist", we mean $\exists \epsilon > 0$, $\forall \delta > 0$, $\exists x_1, x_2$, $0 < |x_1 - a| < \delta$, $0 < |x_2 - a| < \delta$ and $|f(x_1) - f(x_2)| > \epsilon$. In particular, if f jumps at a, $\lim_{x\to a} f(x)$ does not exist.

The proof of some limit laws from definition:

Example 1: prove that $\lim_{x\to c} f(x)g(x) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$.

Let $A = \lim_{x\to c} f(x)$, $B = \lim_{x\to c} g(x)$. $\forall \epsilon > 0$, choose $\epsilon_1 > 0$ such that $\epsilon_1^2 + \epsilon_1(|A| + |B|) < \epsilon$, choose $\delta > 0$ such that $0 < |x-c| < \delta$ implies $|f(x) - A| < \epsilon_1$ and $|g(x) - B| < \epsilon_1$, then $0 < |x-c| < \delta$ implies $|f(x)g(x) - AB| < \epsilon$.

Example 2: prove the Sandwich theorem: $f \leq g \leq F$, $\lim_{x\to c} f(x) = \lim_{x\to c} F(x) = L$, then $\lim_{x\to c} g(x)$ exists and equals L.

$\forall \epsilon > 0$, choose $\delta > 0$ such that $0 < |x-c| < \delta$ implies $|f(x) - L| < \epsilon$ and $|F(x) - L| < \epsilon$, then $0 < |x - c| < \delta$ implies $|g(x) - L| < \epsilon$.

Exercises on the concept of limit:

1. $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist because it

(a) is unbounded.
(b) jumps.
(c) oscillates too much.

[c]

2. $\lim_{x\to 0} x \sin\frac{1}{x}$

(a) does not exist.
(b) =0.
(c) =1.

[a, by Sandwich theorem.]

3. Calculate $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sin x}$.[1/2]

4. Calculate $\lim_{x\to 0}(\ln(\frac{1}{x}+1) - \ln(\frac{1}{x}-1))$. [0]

5. Calculate $\lim_{x\to 0} \frac{\sin(1+x)-\sin(1-x)}{x}$. $[2\cos 1]$

2.4 One-sided limit

$\lim_{x\to c^-} f(x) = L$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x \in (c-\delta, c) \implies |f(x) - L| < \epsilon$.
$\lim_{x\to c^+} f(x) = L$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x \in (c, c+\delta) \implies |f(x) - L| < \epsilon$.

Relation with limit:

If $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$.
If $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$, then $\lim_{x\to c} f(x) = L$
If $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x) = L$, then $\lim_{x\to c} f(x)$ does not exist. (And f *jumps* at c.)

All the basic laws of limit (e.g. the Sandwich theorem) can be adapted to work on one-sided limit.

We can use one-sided limit to show $\lim_{x\to 0} \frac{\sin x}{x} = 1$ as follows:
Step 1: If f is even, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$
Step 2: $\frac{\sin x}{x}$ is even.

Step 3: Show that $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$ with a geometric argument and the Sandwich theorem.

2.5 Continuity

We call f continuous at c if $\lim_{x\to c} f(x) = f(c)$, left continuous if $\lim_{x\to c^-} f(x) = f(c)$, right continuous if $\lim_{x\to c^+} f(x) = f(c)$.

f is called continuous if it is continuous everywhere in its domain.

Geometric meaning: if f is continuous on interval I, then the graph of f restricted on I is connected.

Properties: If f is continuous at c, $\lim_{x\to a} g(x) = c$, then $\lim_{x\to a} f(g(x)) = f(c)$. Specifically, the composition of continuous functions are continuous.

Rational functions, x^a, $\sin x$, $\cos x$, e^x, $\ln x$ are all continuous.

Sum, difference, product, quotient, power, roots preserve continuity.

Intermediate Value Theorem: if f is continuous on interval I, $\exists a, b \in I$ such that $f(a) < c < f(b)$, then there exists $x \in I$ such that $f(x) = c$.

Examples: (1) Find all x such that $f(x) = [x]$ (floor of x) is continuous, left continuous, right continuous.

(2) Show that $\cot x$ is continuous.

(3) True or false: you were once exactly 3 ft tall. (use Intermediate Value Theorem)

(4) We can approximate e^2 with 2.718^2 because (a) e^x is continuous (b) x^2 is continuous. ((b), because x^2 is continuous implies that x^{*2} is close to x^2 if x^* is close to x.)

(5) Find all x where $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$ is continuous. (0)

Further exercises:

(1) Show that $f(x) = |x|$ is continuous. (Hint: use definition to show continuity at 0.)

(2) Show that polynomial $x^{10001} + 56x^{345} + 1000005x^2 + x + 345680907$ has a real root. (Hint: use Intermediate Value Theorem.)

Continuous extension: If f is not defined on c, but $\lim_{x\to c} f(x)$ exists, we can extend the function f to c by defining a new function $F(x) = \begin{cases} f(x), & x \in D(f) \\ \lim_{x\to c} f(x), & x = c \end{cases}$. When defining exponential functions we used the idea of continuous extension.

(3) Assuming the temperature distribution is continuous, show that there are 2 diametrically opposite points on the equator with the same temperature at the same time. (Hint: Intermediate Value Theorem.)

Answers to the problems in the Quantifier worksheet

(b) f is bounded on any finite interval. Example of a function that satisfies (b): $f_1(x) = x$. Example of a function that doesn't satisfy (b): $g_1(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{x}, & x \neq 0 \end{cases}$.

(c) f is constant on an open interval containing x_0. Example of a function that satisfies (b): $f_2(x) = 0$. Example of a function that doesn't satisfy (b): $g_2(x) = x$.

(d) f is bounded. Example of a function that satisfies (b): $f_3(x) = \sin x$. Example of a function that doesn't satisfy (b): $g_3(x) = x$.

(e) The preimage of any finite interval under f is bounded. Example of a function that satisfies (e): $f_4(x) = x$. Example of a function that doesn't satisfy (b): $g_4(x) = \sin x$.

2.6 Limits Involving Infinity

We say $\lim_{x\to c} f(x) = L$, if $f(x)$ can be arbitrary close to L for all $x \neq c$ that are sufficiently close to c. (This is another way of stating the $\epsilon - \delta$ definition.)

We say $\lim_{x\to\infty} f(x) = L$, if $f(x)$ can be arbitrary close to L for all x that are sufficiently large.

We say $\lim_{x\to-\infty} f(x) = L$, if $f(x)$ can be arbitrary close to L for all x such that $-x$ are sufficiently large.

All limit laws we learned earlier are true for $\lim_{x\to\infty}$ and $\lim_{x\to-\infty}$.

Example: $\lim_{x\to\infty}(\ln(x+1) - \ln(x-1))$.

We say $\lim_{x\to c} f(x) = \infty$, if $f(x)$ can be arbitrary large for all $x \neq c$ that are sufficiently close to c.

We can define infinite one-sided limit, infinite limit as $x \to \pm\infty$ similarly.

Example: Calculate $\lim_{x\to 0^+} \tan^{-1}(\frac{1}{x})$. Here $\tan^{-1}$ is the arctangent function, which is the inverse of $f(x) = \tan(x)$, $-\pi/2 < x < \pi/2$.

Horizontal, Oblique & Vertical Asymptotes: horizontal & vertical asymptotes are just ways of saying the existence of finite limit as x goes to infinity, or the existence of infinite limit as x approach some finite number. We say the graph of f has an oblique asymptote $y = kx + b$, if $f(x) - kx - b$ has limit 0 as $x \to \pm\infty$.

Example: $f(x) = \frac{x^2+1}{x} = x + \frac{1}{x}$.

Review for Chapter 1 & 2

- Definition of function, domain, range, natural domain
- Definition of odd/even/increasing/decreasing functions
- Graph of a function
- Operations on functions: sum, difference, product, quotients, composition
- The shifting and scaling of the graph of a function
- Definition of trigonometric functions, exponential functions and logarithms, identities involving these functions
- One-to-one functions and their inverse
- Definition of inverse trigonometric functions
- Limit laws, Sandwich Theorem
- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- Classification of the cases where limit does not exist: "unbounded", "jumps" and "oscillates too much"
- Relationship between limit and one-sided limits
- Continuity and Intermediate Value Theorem
- Limit involving infinity
- Horizontal, Oblique & Vertical Asymptotes
- Average and Instantaneous rate of change

Review Questions:

1. What is the relationship between the graph of $\tan(2x)$ and $\tan(x)$? Write $\tan(2x)$ in terms of $\sin(x)$ and $\cos(x)$, and calculate $\lim_{x\to\pi/3} \tan(2x)$.

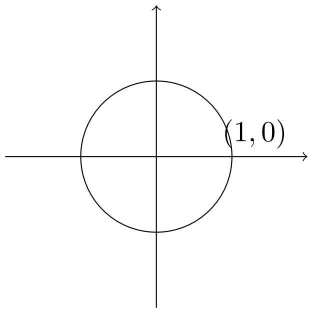
2. $f(x) = \sin\frac{1}{x}$ is an (a) even function. (b) odd function. (c) increasing function. (d) decreasing function.

3. Show that there is a x such that $x^2 = e^x$.

4. Find b such that $f(x) = \begin{cases} b, & x = 0 \\ e^{-\frac{1}{x^2}}, & x \neq 0 \end{cases}$ is continuous.

Trigonometric functions & identities:

The *unit circle* is the circle centered at origin with radius 1.



In mathematics, we measure angles by *radian*, i.e. the length of the arc on the unit circle that subtend the angle from the origin. Given any real number s, consider the arc on the unit circle starting at $(1, 0)$ and has length s. The y-coordinate and x-coordinate of this arc is defined as $\sin s$ and $\cos s$.

By definition, sin is odd and cos is even, both have period 2π, and $\sin^2 x + \cos^2 x = 1$.

Other trigonometric functions are defined as follows: $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$.

Other trigonometric identities you need to remember: $\sin(a+b) = \sin a \cos b + \sin b \cos a$, $\cos(a + b) = \cos a \cos b - \sin a \sin b$, $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$, $\sin 2x = 2 \sin x \cos x$.

Inverse functions: If for any $a, b \in D(f)$, $a \neq b$ implies $f(a) \neq f(b)$, we call f one-to-one. If f is one-to-one, define function f^{-1} on $f(D(f))$ as: $f^{-1}(y) = x$ if and only if $f(x) = y$. We have: for any $x \in D(f)$, $f^{-1}(f(x)) = x$; for any $y \in f(D(f))$, $f(f^{-1}(y)) = y$. We call f^{-1} the *inverse function* of f. To graph f^{-1}, do a reflection of the graph of f by the line $x = y$.

Secant line and tangent line: A *secant line* of the graph of a function f is a line that passes through two points $(a, f(a))$ and $(b, f(b))$, such that $a \neq b$. The slope of the secant line is called the *average rate of change* of f between a and b. When we fix one of the two points, say fixing a, and let b approaches a, if the secant line converges to a line l_a, then l_a is called the *tangent line* of the graph of f at point $(a, f(a))$. The slope of the tangent line is called the *instantaneous rate of change* of f at a.

Exponential and Logarithm: Exponential functions are functions of the form $f(x) = a^x$, where a is a positive real number. The number e is defined by the property that the tangent line of the graph of $y = e^x$ at $(0, 1)$ has slope 1. $\ln x$ is the inverse function of e^x.

Relation between limit and continuity f is continuous at c if (i) the limit of f at c exists, and (ii) this limit equals $f(c)$. The existence of limit at a point c does not depend on the value of $f(c)$, but continuity at c does.

Finding Asymptotes To find horizontal or vertical asymptotes, calculate the limit. To find oblique asymptote of a rational function, write it into the form of $ax + b + f(x)$ such that $\lim_{x \to \pm\infty} f(x) = 0$.

Graphing with computer, ϵ and δ: These are not required topics & will not appear in Prelim 1.

Exercises:

1. Write $\cos(4x)$ in terms of $\cos x$. How about $\cos(5x)$?

2. Find the inverse function of $f(x) = \ln(x+1) - \ln(x-1)$. What is its domain and range?

3. Find the tangent line of $y = \sqrt{x}$ at $(1, 1)$.

3.1,3.2 Derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Left & right hand derivative: $\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$ & $\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$

The existence of derivative implies continuity. The existence of left (right) derivative implies left (right) continuity.

Cases when derivative not exist: discontinuity, corner, cusp, vertical tangent line, etc.

Example: $f(x) = \sqrt[3]{x}$, calculate f'. (hint: use $a^3 - b^3 = (a-b)(a^2+ab+b^2)$)

When the derivate of f exist for all $x \in A$, we call f *differentiable* on A.

Example: $\frac{1}{\sqrt{x^2+1}}$.

Additional Example: $f(x) = \begin{cases} 0, & x = 0 \\ x^3 \sin\frac{1}{x}, & x \neq 0 \end{cases}$ at $x = 0$. (hint: use sandwich theorem)

3.3 Differentiation Rules, n-th Derivative

The function f itself is the 0-th derivative of f.

The n-th derivative of f, denoted as $f^{(n)}$, is the derivative of the $n-1$-th derivative of f.

$f^{(1)}$ is often written as f', $f^{(2)}$ as f'', $f^{(3)}$ as f''' etc.

Geometric meaning: f increasing implies $f' > 0$. f decreasing implies $f' < 0$. f reaches local maximum or minimum at x implies $f'(x) = 0$. f is concave up (or convex in some books) implies $f'' > 0$. f is concave down (or concave in some books) implies $f'' < 0$.

Example: $(\sin x)' = \lim_{h\to 0} \frac{\sin h \cos x + \sin x \cos h - \sin x}{h} = \lim_{h\to 0}(\cos x \cdot \frac{\sin h}{h}) + \lim_{h\to 0}(\sin x \cdot \frac{\cos h - 1}{h}) = \cos x + \lim_{h\to 0}(\sin x \cdot \frac{2\sin^2\frac{h}{2}}{(\frac{h}{2})^2} \cdot \frac{h}{4}) = \cos x + \sin x \cdot 2 \cdot 0 = \cos x$.

Rules for derivatives

a and b are constants, then $(af+bg)' = af' + bg'$.

$(fg)' = f'g + fg'$.

$(\frac{f}{g})' = \frac{f'g - g'f}{g^2}$.

Chain Rule: $(f(g(x)))' = g'(x)f'(g(x))$.

Inverse function rule: $(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$.

$(x^a)' = ax^{a-1}$, $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(e^x)' = e^x$, $(\ln x)' = \frac{1}{x}$,

$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$, $(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$, $(\tan^{-1} x)' = \frac{1}{1+x^2}$.

Exercises:

$(\tan x)'$

$(\frac{e^x}{x^2+1})'$

$f(x) = \begin{cases} 1, & x = 0 \\ \frac{\sin x}{x}, & x \neq 0 \end{cases}$, find f'

Proof that $f'(0) = 0$: $f'(0) = \lim_{h\to 0} \frac{\frac{\sin h}{h} - 1}{h} = \lim_{h\to 0} \frac{\sin h - h}{h^2}$. $\frac{\sin h - h}{h^2}$ is an odd function, so we only need to show $\lim_{h\to 0^+} \frac{\sin h - h}{h^2} = 0$. Use Sandwich Theorem. When $h > 0$, $\sin h \le h \le \tan h$, so $\frac{\sin h - \tan h}{h^2} \le \frac{\sin h - h}{h^2} \le 0$. $\lim_{h\to 0+} 0 = 0$, $\lim_{h\to 0^+} \frac{\sin h - \tan h}{h^2} = \lim_{h\to 0^+}(\frac{\sin h}{h} \cdot \frac{\cos h - 1}{h \cos h}) = \lim_{h\to 0^+} \frac{\sin h}{h} \cdot \lim_{h\to 0^+} \frac{1}{\cos h} \cdot \lim_{h\to 0^+} \frac{-2\sin^2 \frac{h}{2}}{(\frac{h}{2})^2} \cdot \lim_{h\to 0^+} \frac{h}{4} = 0$, so

$\lim_{h \to 0^+} \frac{\sin h - h}{h^2} = 0$. q.e.d.

$$g(x) = \begin{cases} 0, & x = 0 \\ x \sin \frac{1}{x}, & x \neq 0 \end{cases}, \text{ find } g'$$

$(x^{\sin x})'$

3.4 Application of Derivatives

Speed, acceleration, jerk (i.e. the first, second, and third order derivative of the displacement with regards to time)

Margin in economics.

Sensitivity to change.

3.5-3.6, 3.8-3.9 Proof of the rules of derivatives

(i) a and b are constants, then $(af + bg)' = af' + bg'$.
(ii) $(fg)' = f'g + fg'$.
(iii) $(\frac{f}{g})' = \frac{f'g - g'f}{g^2}$.
(iv) Chain Rule: $(f(g(x)))' = g'(x)f'(g(x))$.
(v) Inverse function rule: $(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$.
(vi) $(x^a)' = ax^{a-1}$
(vii) $(\sin x)' = \cos x$
(viii) $(\cos x)' = -\sin x$
(ix) $(e^x)' = e^x$
(x) $(\ln x)' = \frac{1}{x}$
(xi) $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$
(xii) $(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$
(xiii) $(\tan^{-1} x)' = \frac{1}{1+x^2}$.

Proof. (i) follows directly from the laws of limit.

(ii) $(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = \lim_{h\to 0} \frac{(f(x+h)g(x+h)-f(x)g(x+h))+(f(x)g(x+h)-f(x)g(x))}{h} = f'g + fg'$.

(iii) $\frac{f}{g} = f \cdot g^{-1}$, hence it follows from (ii), (iv) and (vi).

(iv) is based on an equivalent definition of derivative as linear approximation.

So, $f(g(x+h)) = f(g(x) + g'(x)h + o(h)) = f(g(x)) + f'(g(x))g'(x)h + f'(g(x))o(h) + o(g'(x)h + o(h)) = f(g(x)) + f'(g(x))g'(x)h + o(h)$.

(v) follows from (iv) and the fact that $f(f^{-1}(x)) = x$.

(vi) for positive x, $x^a = e^{\ln x^a} = e^{a \ln x}$, hence the formula follows from (iv), (ix) and (x). For negative x, use $(-x)^a = (-1)^a x^a$.

(vii) $\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0} \frac{\sin x(\cos h-1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} = \lim_{h\to 0} \frac{\sin h(-2\sin^2 \frac{h}{2})}{h} + \cos x = \cos x$.

(viii) follows from (vii) and the fact that $\cos(x) = \sin(\pi/2 - x)$.

(ix) $(e^x)' = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0} \frac{e^h-1}{h} = e^x (e^x)'|_{x=0} = e^x$.

(x) follows from (ix) and (v).

(xi) follows from (v) and (vii).

(xii) follows from (v) and (viii).

(xiii) follows from (iii), (v), (vii), and (viii). $\square$

Example: $(\cot^{-1} x)'$

$(|x|^a)'$

$(\log_{1+x^2}(1+x^4))'$

3.11 The Proof of the Chain Rule

Theorem 1. *$f'(x) = a$ if and only if there exists a function ϵ defined around 0 such that $f(x+h) = f(x) + ah + \epsilon(h)h$, and $\lim_{h\to 0} \epsilon(h) = 0$.*

Remark 1. The function ϵ depends on f and x.

Proof. Suppose $f'(x) = a$, then $f(x+h) = f(x) + ah + f(x+h) - f(x) - ah = f(x) + ah + [\frac{f(x+h)-f(x)}{h} - a]h$, and $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - a = f'(x) - a = 0$, so we can let $\epsilon(h) = \frac{f(x+h)-f(x)}{h} - a$ when $h \neq 0$ and $\epsilon(h) = 0$ when $h = 0$, and have $f(x+h) = f(x) + ah + \epsilon(h)h$.

Suppose there is a function ϵ such that $f(x+h) = f(x) + ah + \epsilon(h)h$, and $\lim_{h\to 0} \epsilon(h) = 0$, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{ah+\epsilon(h)h}{h} = a + \lim_{h\to 0} \epsilon(h) = a$. $\square$

Now we can prove the chain rule using the previous theorem:

$f(g(x+h)) = f(g(x) + g'(x)h + \epsilon_g(h)) = f(g(x)) + f'(g(x))(g'(x)h + \epsilon_g(h)h) + \epsilon_f(g'(x)h + \epsilon_g(h))(g'(x)h + \epsilon_g(h)h) = f(g(x)) + f'(g(x))g'(x)h + [f'(g(x))\epsilon_g(h) + \epsilon_f(g'(x)h + \epsilon_g(h)h)(g'(x) + \epsilon_g(h))]h$, where ϵ_f is the ϵ function corresponding to f and $g(x)$ and ϵ_g is the ϵ function corresponding to g and x. $\lim_{h\to 0}[f'(g(x))\epsilon_g(h) + \epsilon_f(g'(x)h + \epsilon_g(h)h)(g'(x) + \epsilon_g(h))] = 0$, so $(f(g(x)))' = f'(g(x))g'(x)$.

Remark 2. The geometric meaning of ϵ is the difference between the slope of the secant line and the tangent line.

3.7 Derivative of implicit function

Suppose function $y(x)$ satisfies $(y(x))^2 + xy(x) + x^2 = 3$, $y(1) = 1$, find $y'(1)$, $y''(1)$. (Hint: use Chain Rule)

Proof of the Rules of Differentiation:

(1) a and b are constants, then $(af+bg)' = af' + bg'$
(2) $(\sin x)' = \cos x$
(3) $(e^x)' = e^x$
(4) $(fg)' = f'g + fg'$
(5) Chain Rule: $(f(g(x)))' = g'(x)f'(g(x))$
(6) Inverse function rule: $(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$
(7) $(\cos x)' = -\sin x$
(8) $(\ln x)' = \frac{1}{x}$
(9) $(x^a)' = ax^{a-1}$
(10) $(\frac{f}{g})' = \frac{f'g-g'f}{g^2}$
(11) $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$
(12) $(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$
(13) $(\tan^{-1} x)' = \frac{1}{1+x^2}$

Proof. (1) follows directly from the limit laws.

(2) $\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0} \frac{\sin x(\cos h-1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} = \lim_{h\to 0} \frac{\sin x(-2\sin^2 \frac{h}{2})}{h} + \cos x = -2\sin x \lim_{h\to 0}(\frac{\sin\frac{h}{2}}{\frac{h}{2}})^2 \cdot \frac{h}{4} + \cos x = \cos x$. Here, we used the following 2 trigonometric identities: $\sin(a+b) = \sin a\cos b + \cos a \sin b$, and $\cos c = 1 - 2\sin^2 \frac{c}{2}$.

(3) Let $f(x) = e^x$, then by the definition of number e, $f'(0) = 1$. $f'(x) = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0} \frac{e^h-1}{h} = e^x f'(0) = e^x$.

(4) $(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = \lim_{h\to 0} \frac{(f(x+h)g(x+h)-f(x)g(x+h))+(f(x)g(x+h)-f(x)g(x))}{h} = f'g + fg'$.

(5)

Lemma. *$f'(x) = a$ if and only if there exists a function ϵ defined around 0 such that $f(x+h) = f(x) + ah + \epsilon(h)h$, and $\lim_{h\to 0}\epsilon(h) = 0$.*

Proof. Suppose $f'(x) = a$, then $f(x+h) = f(x) + ah + f(x+h) - f(x) - ah = f(x) + ah + [\frac{f(x+h)-f(x)}{h} - a]h$, and $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - a = f'(x) - a = 0$, so if we define $\epsilon(h) = \begin{cases} \frac{f(x+h)-f(x)}{h} - a, & h \neq 0 \\ 0, & h = 0 \end{cases}$, ϵ satisfies all the requirements in the lemma.

On the other hand, suppose there is a function ϵ such that $f(x+h) = f(x) + ah + \epsilon(h)h$, and $\lim_{h\to 0}\epsilon(h) = 0$, then $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{ah+\epsilon(h)h}{h} = a + \lim_{h\to 0}\epsilon(h) = a$. $\square$

Remark. Geometrically, the function ϵ is the difference between the slope of the secant line between $(x, f(x))$ and $(x+h, f(x+h))$ and the slope of the tangent line at $(x, f(x))$, and the lemma is saying that the existence of limit is equivalent to the convergence of the secant line as $h \to 0$. ϵ depends on f and x.

Now we can prove the chain rule using the previous lemma:

$f(g(x+h)) = f(g(x)+g'(x)h+\epsilon_g(h)) = f(g(x))+f'(g(x))(g'(x)h+\epsilon_g(h)h)+\epsilon_f(g'(x)h+\epsilon_g(h))(g'(x)h+\epsilon_g(h)h) = f(g(x))+f'(g(x))g'(x)h+[f'(g(x))\epsilon_g(h)+\epsilon_f(g'(x)h+\epsilon_g(h)h)(g'(x)+\epsilon_g(h))]h$, where ϵ_f is the ϵ function corresponding to f and $g(x)$ and ϵ_g is the ϵ function corresponding to g and x. $\lim_{h\to 0}[f'(g(x))\epsilon_g(h) + \epsilon_f(g'(x)h + \epsilon_g(h)h)(g'(x) + \epsilon_g(h))] = 0$, so $(f(g(x)))' = f'(g(x))g'(x)$.

(6)-(13) follows from (1)-(5). □

Examples:

1. Function $y(x)$ satisfy $x^2 + yx + y^2 = 3$, $y(1) = 1$. Calculate $y'(1)$ and $y''(1)$.

Solution: By assumption, $F(x) = x^2 + y(x)x + (y(x))^2 = 3$, so $F' = F'' = 0$. $F' = 2x + y + y'x + 2y'y = 0$ so $y'(1) = -1$. $F'' = 2 + y' + y''x + y' + 2y''y + 2y'y' = 0$, so $3y'' + 2 = 0$, $y'' = -2/3$. (Sorry I made a mistake on this problem in the morning.)

2. $(\csc^{-1})''$

Solution 1: If $y = \csc^{-1}(x)$ then $x = \csc y = \frac{1}{\sin y}$, so $\sin y = \frac{1}{x}$, $y = \sin^{-1}\frac{1}{x}$. Now use the chain rule, $(\csc^{-1}(x))' = (\sin^{-1}\frac{1}{x})' = \frac{-\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}} = -\frac{1}{|x|\sqrt{x^2-1}}$, and $(\csc^{-1}(x))'' = \frac{(|x|\sqrt{x^2-1})'}{x^2(x^2-1)} = \frac{\frac{x}{|x|}\sqrt{x^2-1}+\frac{|x|x}{\sqrt{x^2-1}}}{x^2(x^2-1)} = \frac{|x|(2x^2-1)}{x^3(x^2-1)^{\frac{3}{2}}}$.

Here we do not need to deal with $x = 0$ because 0 is not in the domain of $\csc^{-1}$.

Solution 2: Use the inverse rule, $(\csc^{-1}(x))' = \frac{1}{\csc'(\csc^{-1} x)} = -\frac{1}{\frac{\cos(\csc^{-1} x)}{\sin^2(\csc^{-1} x)}} = -\frac{\sin^2(\csc^{-1} x)}{\sqrt{1-\sin(\csc^{-1} x)}}$. Because if $u = \csc^{-1}(x)$ then $x = \frac{1}{\sin u}$, so $\sin(\csc^{-1} x) = \sin u = \frac{1}{x}$, $(\csc^{-1}(x))' = \frac{-\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}}$. Then do the same calculation as in Solution 1.

3. $(\tan^{-1}(\ln(x)))'$

Solution: By chain rule, $\tan^{-1}(\ln(x)))' = \frac{\frac{1}{x}}{1+(\ln(x))^2}$.

Other topics on derivative

(1) The *normal line* of a plane curve γ through a point P on γ is the line passes through P and is orthogonal to the tangent line of γ at P.

Example: Find the tangent line and normal line of $y^2 + xy + x^2 = 3$ at $(1, 1)$.

Example: Exercise 52

(2) Calculate the composition of trigonometric and inverse trigonometric function:

Example: $\cos(\sec^{-1}(x))$: if $y = \sec^{-1} x$, then $x = \sec y = \frac{1}{\cos y}$, so sec is the composition of $\frac{1}{x}$ and cos. We know that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, so $\sec^{-1}(x) = \cos(\frac{1}{x})$, $\cos(\sec^{-1}(x)) = \frac{1}{x}$.

Example: $\tan(\sin^{-1} x)$.

(3) Calculate the derivative of products. $(\prod_i f_i)' = (\sum_i \frac{f_i'}{f_i}) \prod_i f_i$. (proved by induction or by applying ln on both sides)

Calculate $((1 + x^2)(1 + x^4)(1 + x^6)(1 + x^8))'$

(4) Use derivative to calculate limit:

Calculate $\lim_{x\to 0}(1 + x)^{\frac{1}{x}}$

Calculate $\lim_{x\to 0} \frac{\sin x}{\sin 3x}$

Homework from 3.1-3.7

3.1.18, 3.3.64: the slope of the tangent line is the derivative.

3.1.32: the "rate of change" is the derivative.

(*)3.1.35: calculate the derivative of $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$ by definition.

3.2.8, 3.2.12, 3.3.2, 3.3.6, 3.3.36, 3.5.2, 3.5.6, 3.5.18, 3.5.26, 3.6.6, 3.6.18, 3.6.28, 3.6.34, 3.6.55, 3.6.82: calculation of derivative with the rules.

3.3.54: use differentiation rules.

3.4.10: maximum iff derivative is 0.

(*)3.2.28, 3.2.30, 3.4.10: sketching derivative.

(*)3.5.57: use the limit of $\sin(x)/x$.

(*)3.7.16, 3.7.20, 3.7.30: implicit differentiation.

The number e

Theorem 2. $e = \lim_{x\to 0}(1+x)^{1/x}$

Proof. $\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0}(e^{\frac{\ln(1+x)}{x}}) = e^{\lim_{x\to 0}\frac{\ln(1+x)-\ln(1)}{x}} = e^{(\ln)'(1)} = e$. $\square$

Remark 3. We will see later or in Calc 2 that $e = \sum_n \frac{1}{n!}$.

Word problems with differentiation

Steps:
1. *Name* the variables.
2. Write down the given *numbers* (if there are any).
3. Write down the relation between variables as *equations*.
4. *Differentiate* (often w.r.t. time).
5. Find the answer.

Example 0: A point moves on the unit circle $x^2+y^2=1$. If $x(t_0)=0.6$, $y(t_0)=0.8$, $x'(t_0)=1$, $x''(t_0)=0$, find $y'(t_0)$ and $y''(t_0)$.

Example 1: Point A is moving on the positive y-axis downwards with speed 1, and is currently at $(0,1)$, point B is moving on the positive x-axis and the distance between them is always $\sqrt{2}$. Find the speed of point B and the midpoint of line segment AB.

Example 2: (from Kepler's laws to universal gravitation) The orbit of a planet around its sun is $\rho(a+\sin\theta)=1$, where $a>1$.
(1) Find the relationship between $\frac{d\rho}{dt}$ and $\frac{d\theta}{dt}$.
(2) If we further assume $\rho^2\frac{d\theta}{dt}=1$, write $\frac{d^2}{dt^2}(\rho\cos\theta)$ as a function of ρ and θ.
(3) Do the same for $\rho\sin\theta$.
(4) Write the acceleration of the planet as a function of ρ and θ.

Solution for Example 2:

(1) $\rho'(a+\sin\theta)+\rho\theta'\cos\theta=0$

(2) Because $\rho^2\theta'=1$, $\rho'(a+\sin\theta)+\frac{\cos\theta}{\rho}=0$, $\rho'=-\cos\theta$. Hence, $(\rho\cos\theta)'=\rho'\cos\theta-\rho\theta'\sin\theta=-\cos^2\theta-\frac{\sin\theta}{\rho}=\cos^2\theta-(a+\sin\theta)\sin\theta=1-a\sin\theta$, $(\rho\cos\theta)''=(1-a\sin\theta)'=-\frac{a}{\rho^2}\cos\theta$.

(3) Similar to (2), we have $(\rho\sin\theta)''=-\frac{a}{\rho^2}\sin\theta$.

(4) From (2), (3), we know the acceleration is $\frac{a}{\rho^2}$ in the direction towards the origin.

3.11 Standard linear approximation and differential

If f is differentiable at a, we call $f(a+h)\approx f(a)+hf'(a)$ the *standard linear approximation*. We can also let $x=a+h$ and write it as $f(x)\approx f(a)+(x-a)f'(a)$.

Formula like $\sin'(x) = \cos x$ can be written as $d\sin x = \cos x dx$. Here the symbol d followed by a function is called a *differential.*

4.1 Critical points

a is called a *critical point* of f if $f'(a) = 0$, **or** a **is in the interior of** $D(f)$ **and** $f'(a)$ **is undefined**.

Remark 4. **Sorry, I made a mistake in class in the definition of critical points.**

Exercises:

1. Find the domain, range, and derivative of $y = \ln(\ln x)$.

2. Find the derivative of $\sin(\cot^{-1} x)$.

3. Let $f(x) = \begin{cases} x^2 \sin(\ln|x|), & x \neq 0 \\ 0, & x = 0 \end{cases}$. Find all its critical points.

We call a a *local maximum* of f, if there is an open interval I containing a such that $f(a) \geq f(x)$ for all x in $I \cap D(f)$. *Local minimum* is defined similarly.

Theorem 3. *If f is differentiable at a, a is a local maximum or local minimum, then $f'(a) = 0$.*

Theorem 4. *If f is continuous and is defined on a finite closed interval $[a, b]$, then it reaches its maximum and minimum somewhere on $[a, b]$.*

The above theorem follows from the following property of real numbers:

Lemma 5. *Let $\{x_n\}$ be an infinite sequence of real numbers in a finite closed interval I, there must be some $y \in I$ such that any open interval containing y also contains infinitely many x_n.*

Examples:

Find the maximum and minimum of the following functions:

1. $\sqrt{x - x^2}$

2. $e^x + e^{-x}$

4.2, 4.3 Mean Value Theorem and its applications

Theorem 6. *1. If f is continuous on $[a, b]$ and differentiable on (a, b), $f(a) = f(b)$, then there is some $c \in (a, b)$ such that $f'(c) = 0$.*

2. If f is continuous on $[a,b]$ and differentiable on (a,b), then there is some $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Corollary 1. *1. If $f'(x) = 0$ on an interval I then f is constant on that interval.*
2. If $f' = g'$ on an interval I then $f - g$ is constant on that interval.
3. If $f' > 0$ on an interval I then f is increasing on I, If $f' < 0$ on an interval I then f is decreasing on I.

Examples: 1. Show that $e^x - 100x^2$ has at most 3 zeros.

2. Show that $\ln(ab) = \ln a + \ln b$

3. Consider function $f(x) = \begin{cases} x^2 \sin(1/x) + x/10, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x) + 0.1, & x \neq 0 \\ 0.1, & x = 0 \end{cases}$. $f'(0) > 0$ but f is neither increasing nor decreasing on any interval containing 0. Consider the derivative: $2x\sin(1/x) - \cos(1/x) + 0.1$, the first term is small for small x, the second term oscillates between ± 1, so the derivative oscillates between 1.1 and -0.9 as $x \to 0$. In other words, any interval I containing x must have some region where $f' > 0$ and some region where $f' < 0$, hence f can not be increasing or decreasing on I.

4. Show that $\cos x \geq 1 - x^2/2$. Because cos and $1 - x^2/2$ are both even we only need to show it for $x > 0$. Consider $F(x) = \cos x - (1 - x^2/2)$. $F'(x) = -\sin x + x > 0$, so F is increasing when $x > 0$. We also know that $F(0) = 0$, so $F(x) \geq 0$ for all $x \geq 0$, i.e. $\cos x \geq 1 - x^2/2$.

Corollary 2. *(First derivative test) c is a local min (max) if f' changes from positive to negative (negative to positive) at c.*

Example: $f(x) = x^3 - x$. $f' = 3x^2 - 1$, $f' > 0$ when $x < -\frac{\sqrt{3}}{3}$ or $x > \frac{\sqrt{3}}{3}$, $f' < 0$ when $-\frac{\sqrt{3}}{3} < x < \frac{\sqrt{3}}{3}$, so $-\frac{\sqrt{3}}{3}$ is a local max and $\frac{\sqrt{3}}{3}$ is a local min.

Exercises: Find the local max/min of the following functions:

1. $x^x e^{-x}$

Solution: $(x^x e^{-x})' = (e^{x\ln x} e^{-x})' = (e^{x\ln x - x})' = (x\ln x - x)' e^{x\ln x - x} = (\ln x + 1 - 1)e^{x\ln x - x} = \ln x e^{x\ln x - x}$. So by the first derivative test, $x = 1$ is a local minimum.

2. $|x^4 - x^2|$

Solution: $|x^4 - x^2| = \begin{cases} x^4 - x^2, & x \geq 1 \text{ or } x \leq -1 \\ x^2 - x^4, & -1 < x < 1 \end{cases}$. By calculating derivative we know that $f(x) = |x^4 - x^2|$ is decreasing on the following intervals: $(-\infty, -1)$, $(-\frac{\sqrt{2}}{2}, 0)$, $(\frac{\sqrt{2}}{2}, 1)$, and increasing on the following intervals: $(-1, -\frac{\sqrt{2}}{2})$, $(0, \frac{\sqrt{2}}{2})$, and $(1, \infty)$. So $0, \pm 1$ are the local minimums and $\pm\frac{\sqrt{2}}{2}$ are the local maximums.

3. $\sin(x) + \sin(x+1)$

Solution 1: $\sin(x) + \sin(x+1) = \sin((x+1/2) - 1/2) + \sin((x+1/2) + 1/2) = 2\sin(x + 1/2)\cos(1/2)$. So it reaches local maximum when $x = \frac{\pi-1}{2} + 2k\pi$, local minimum when $x = \frac{-\pi-1}{2} + 2k\pi$, where $k \in \mathbb{Z}$.

Solution 2: Let $f(x) = \sin(x) + \sin(x+1)$, then $f'(x) = \cos x + \cos(x+1) = \cos x + \cos x \cos 1 - \sin x \sin 1 = (1 + \cos 1)\cos x - \sin 1 \sin x$, hence $f'(x) = 0$ means $\tan x = \frac{1+\cos 1}{\sin 1}$, i.e. $x \in \{x_k\}$ where $x_k = \tan^{-1}\frac{1+\cos 1}{\sin 1} + k\pi$, where $k \in \mathbb{Z}$. $f'(x) = (1 + \cos 1)\cos x - \sin 1 \sin x = \frac{\cos x}{\sin 1} \cdot (\frac{1+\cos 1}{\sin 1} - \tan x)$, so f' is positive between x_{2k-1} and x_{2k}, and negative between x_{2k} and x_{2k+1}, hence x_k is local max when k is even, local min when k is odd.

4. $\frac{1}{5}x^5 - \frac{2}{3}x^3 + x - \frac{1}{17}$

Solution: Let $f(x) = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x - \frac{1}{17}$. $f'(x) = (x^2 - 1)^2 \geq 0$, and $f'(x) > 0$ when $x \neq \pm 1$. So, f is increasing on $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$, which implies that f is increasing on $\mathbb{R}$, hence it has no local min or local max.

CONCAVITY AND SECOND DERIVATIVE TEST

$f'' > 0$ implies that f' is increasing, which we call *"f is concave up"*.

$f'' < 0$ implies that f' is decreasing, which we call *"f is concave down"*.

Places where **tangent line exists (i.e. f' exists or $f' = \infty$) and concavity changes** are called *inflection points*.

By Mean Value Theorem (MVT), concave up implies that any line segment linking two points on the graph of f lies above the graph of f, and concave down implies that any line segment linking two points on the graph of f lies below the graph of f. In other math textbooks "concave up" is often called "convex" and "concave down" is often called "concave".

In the future, when sketching the graph of a function, we need to pay attention to concavity.

Second derivative test: if $f'(a) = 0$ and $f''(a) > 0$ then a is local min, if $f'(a) = 0$ and $f''(a) < 0$ then a is a local max.

Example: sketch the graph of $x^3 - x$.

Sketch the graph of $\tan^{-1} x$.

Sketch the graph of x^x.

L'Hospital's rule

This is a generalization of the argument we used earlier to calculate limits like $\lim_{x\to 0} \frac{\sin x}{\ln(x+1)}$.

Theorem 7. *1. If $\lim_{x\to a^+} f = \lim_{x\to a^+} g = 0$, g and g' are nonzero on some interval $(a, a+d)$, and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = A$ exist, then $\lim_{x\to a^+} \frac{f(x)}{g(x)}$ also exists and equals A.*

2. If $\lim_{x\to a^+} f = \lim_{x\to a^+} g = \infty$, g and g' are nonzero on some interval $(a, a+d)$, and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = A$ exist, then $\lim_{x\to a^+} \frac{f(x)}{g(x)}$ also exists and equals A.

Example: $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$.

$\lim_{x\to 0^+} x^{\sqrt{x}}$ (Hint: rewrite it as $e^{\frac{\ln x}{x^{-1/2}}}$.

Non-example: When $f(x) = x^2 \sin\frac{1}{x}$, $g(x) = x$, $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$, but $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist.

$\lim_{x\to 0} \frac{\sin x - x}{x^3}$. (answer: $-\frac{1}{6}$)

$\lim_{x\to 0}(\frac{1}{\ln(x+1)} - \frac{1}{x})$

Solution: $\lim_{x\to 0}(\frac{1}{\ln(x+1)} - \frac{1}{x}) = \lim_{x\to 0} \frac{x - \ln(x+1)}{x\ln(x+1)} = \lim_{x\to 0} \frac{1 - \frac{1}{x+1}}{\ln(x+1) + \frac{x}{x+1}} = \lim_{x\to 0} \frac{x}{(x+1)\ln(x+1) + x} = \lim_{x\to 0} \frac{1}{1 + \ln(x+1) + 1} = \frac{1}{2}$.

Remark 5. There are many ways to write $\frac{1}{\ln(x+1)} - \frac{1}{x}$ into a quotient, but not all are convenient for the application of l'Hospital rule.

Proof of the L'Hospital's rule:

Lemma 8. *(Cauchy's MVT) If f and g are both continuous on $[a,b]$ and differentiable on (a,b), $g(a) \neq g(b)$, then there is a $c \in (a,b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.*

Proof of Lemma: Use the MVT on $F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x) - g(a))$.

Remark 6. A common technique for applying MVT is to build functions then use MVT on them. For example, if we know $f(0) = f(1) = 1$, $f'' \geq -1$, then $f(x) \leq \frac{1}{8}$. To show that, consider $F(x) = f(x) - \frac{1}{2}x(1-x)$. Now we only need to show that $F \leq 0$. If not, by MVT, there is some $a \in (0,1)$ such that $F(a) > 0$, hence there is some $b \in (0,a)$ such that $F'(b) > 0$, and some $c \in (a,1)$ such that $F'(c) < 0$. Use MVT again, there is some point $m \in (b,c) \subset (0,1)$ such that $F''(m) = f''(m) - (-1) < 0$, a contradiction. It is easy to see that here $\frac{1}{8}$ is the best possible bound.

Now we can prove L'Hospital's rule. The proof of part 1 is in the textbook. The proof of part 2 is as follows:

Use the definition of limit. Given $\epsilon > 0$. Because the limit of f'/g' is A, there is some $\delta > 0$ such that on $(a, a+\delta)$, $|f'/g' - A| < \epsilon/2$. Let $M = \frac{1000}{\epsilon}\max\{|f(a+\delta)|, |g(a+\delta)|\}$, $0 < \delta' < \delta$ such that on $(a, a+\delta')$, $|g| > \frac{2}{\epsilon}\max\{|g(a+\delta)|, |f(a+\delta) - (A-\epsilon)g(a+\delta)|, |f(a+\delta) - (A+\epsilon)g(a+\delta)|\}$. Now for any $x \in (a, a+\delta')$, use Cauchy's MVT on $(a, a+\delta)$, we have $A - \frac{\epsilon}{2} < \frac{f(x)-f(a+\delta)}{g(x)-g(a+\delta)} < A + \frac{\epsilon}{2}$. Hence, $f(x) - f(a+\delta) = A'(g(x) - g(a+\delta))$, where $|A' - A| < \epsilon/2$. Therefore, $\frac{f(x)}{g(x)} = A' + \{\frac{f(a+\delta) - A'g(a+\delta)}{g(x)}\}$, and the assumption on δ' tells us that the term in $\{\}$ is bounded by $\frac{\epsilon}{2}$. q.e.d.

Summary on sketching the graph of a function

When sketching the graph of a function, pay attention to the following: domain, symmetry, critical points, limit points, boundary points, monotonicity, inflection points, concavity, asymptotes, intersection with x and y axis.

When the exact positions of certain points can not be written down explicitly, sketch its approximated position.

Example: sketch the graph of $e^{-\frac{1}{x}}$.

4.6 Applications of max & min

Example 1: Find the maximum volume of right circular cones with surface area S.

Solution: A right circular cone with radius r and height h has volume $V = \frac{1}{3}\pi r^2 h$ and surface area $S = \pi r^2 + \frac{1}{2}\sqrt{r^2+h^2} \cdot 2\pi r = \pi r(r + \sqrt{r^2+h^2})$. Fix S, solve for h with respect to r we get $h = \frac{1}{r} \cdot \sqrt{\frac{S}{\pi}} \cdot \sqrt{\frac{S}{\pi} - 2r^2}$, hence $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\sqrt{\pi S} \cdot r\sqrt{\frac{S}{\pi} - 2r^2}$. Use the first derivative test we know that V reaches its maximum when $r = \frac{1}{2}\sqrt{\frac{S}{\pi}}$, and the maximum is $\frac{S^{\frac{3}{2}}}{3 \cdot 2^{\frac{3}{2}} \pi^{\frac{1}{2}}}$.

Example 2: (maximum sustainable yield) $ky(C-y) - M = 0$, maximize M.

Example 3: (the dual of Example 1) Find the minimal surface area of right circular cones with volume V.

Solution: A right circular cone with radius r and height h has volume $V = \frac{1}{3}\pi r^2 h$ and surface area $S = \pi r^2 + \frac{1}{2}\sqrt{r^2+h^2} \cdot 2\pi r = \pi r(r + \sqrt{r^2+h^2})$. Fix V, solve for h with respect to r we get $h = \frac{3V}{\pi r^2}$, hence $S = \pi r(r + \sqrt{r^2+h^2}) = \pi r(r + \sqrt{r^2 + \frac{9V^2}{\pi^2 r^4}})$. Use the first derivative test we know that S reaches its minimum when $r = (\frac{9V^2}{8\pi^2})^{\frac{1}{6}}$, and the minimum is $3^{\frac{2}{3}} \cdot 2 \cdot \pi^{\frac{1}{3}} \cdot V^{\frac{2}{3}}$.

4.7 Newton's Method

$x_{n+1} = x_n - f(x_n)/f'(x_n)$

Situations in which it converges slowly or does not converge: non-existence or non-continuous derivative, the starting point is too far away, $f'(x^*) = 0$ where x^* is the root we are trying to find, etc.

4.8 Antiderivative

$F' = f$, then F is called an antiderivative of f. The collection of all antiderivatives of f is called the indefinite integral of f: $\int f(x)dx$.

By MVT, if f is defined on an interval, two antiderivatives of f must differ by a constant. So, if $F' = f$ then $\int f dx = F + C$.

Example: $\int (x+1)dx$, $\int \sec x \tan x + e^x dx$, $\int \sin 2x + \cos x dx$.

Terminal velocity: if $\frac{dv}{dt} = g - kv^2$, then $t = \int \frac{1}{g-kv^2} dv$.

Unlike differentiation, it is hard to find the antiderivative of elementary function symbolically. The computer algebra system Maxima implemented an algorithm for symbolic integration. You can read the source code (in Common Lisp) here:
http://sourceforge.net/p/maxima/code/ci/master/tree/src/risch.lisp

5.1 Area, upper/lower/midpoint sums

The concept of integration is based on our intuition about area.

The problem of finding displacement from velocity, and finding the average of a function, can be reduced to finding area.

To estimate the area below a function f, which is defined on a finite interval $[a,b]$, we can decompose $[a,b]$ into n subintervals of equal length, and define the following concepts:

(1) Upper sum: $\frac{b-a}{n}\sum_{k=0}^{n-1} M_k$, where M_k is the maximum of f on $[a+\frac{k(b-a)}{n}, a+\frac{(k+1)(b-a)}{n}]$. When the maximum does not exist, use the smallest upper bound.
(2) Lower sum: $\frac{b-a}{n}\sum_{k=0}^{n-1} m_k$, where m_k is the minimum of f on $[a+\frac{k(b-a)}{n}, a+\frac{(k+1)(b-a)}{n}]$. When the minimum does not exist, use the greatest lower bound.
(3) Mid-point sum: $\frac{b-a}{n}\sum_{k=0}^{n-1} f(a+\frac{(k+\frac{1}{2})(b-a)}{n})$.
(4) Left-end-point sum: $\frac{b-a}{n}\sum_{k=0}^{n-1} f(a+\frac{k(b-a)}{n})$.
(5) Right-end-point sum: $\frac{b-a}{n}\sum_{k=0}^{n-1} f(a+\frac{(k+1)(b-a)}{n})$.

If the upper sum and lower sum converges to a common limit S as $n\to\infty$, S must be the area.

Example: find the area below $f(x)=x$, $x\in[0,1]$.

Example: (Archimedes, c. 250 BCE) find the area of a unit disc from the area of regular n-gons.

Example: find the area below $f(x)=\sin x$, $x\in[0,\pi/2]$.

Solution: The n-th lower sum is $\frac{\pi}{2n}\sum_{k=0}^{n-1}\sin\frac{k\pi}{2n} = \frac{\pi}{2n}\frac{1}{\sin\frac{\pi}{4n}}\sin\frac{\pi}{4n}\sum_{k=0}^{n-1}\sin\frac{k\pi}{2n} = \frac{\frac{\pi}{2n}}{\sin\frac{\pi}{4n}}\cdot \sum_{k=0}^{n-1}[\cos(\frac{(k-\frac{1}{2})\pi}{2n})-\cos(\frac{(k+\frac{1}{2})\pi}{2n})] = \frac{\frac{\pi}{2n}}{\sin\frac{\pi}{4n}}\cdot[\sum_{k=0}^{n-1}\cos(\frac{(k-\frac{1}{2})\pi}{2n})-\sum_{k=1}^{n}\cos(\frac{(k-\frac{1}{2})\pi}{2n})] = \frac{\frac{\pi}{2n}}{\sin\frac{\pi}{4n}}\cdot [\cos(-\frac{\pi}{4n})-\cos(\frac{\pi}{2}-\frac{\pi}{4n})]$. When $n\to\infty$, the lower sum converges to 1. A similar calculation will show that the upper sum converges to 1 when $n\to\infty$. Hence, this area is 1.

5.2 The summation symbol

Example: calculate the following:

$\sum_{k=1}^{10} 2k-1$

Let $S=1+3+5+7+\cdots+19$, then $S=19+17+\cdots+3+1$. Add the two equations, we get $2S=(1+19)+(3+17)+\cdots+(19+1)=20\times 10=200$, so $S=100$.

$\sum_{k=0}^{n}(-1)^n$

When n is even, the sum is $1-1+1-1+1-1+\cdots+1=1+(-1+1)+(-1+1)+(-1+1)+\cdots+(-1+1)=1$. When n is odd, the sum is $1-1+1-1+1-1+\cdots+1-1=(1-1)+(1-1)+\cdots+(1-1)=0$.

$\sum_{k=0}^{100}3^{-n}$

Let $S=\sum_{k=0}^{100}3^{-n}$, then $\frac{1}{3}S=\frac{1}{3}\sum_{k=0}^{100}3^{-n}=\sum_{k=1}^{101}3^{-n}$. Hence, $\frac{1}{3}S-S=3^{-101}-3^0=3^{-101}-1$, $S=\frac{3^{-101}-1}{3^{-1}-1}$. In general, when $p\neq 1$, $\sum_{n=0}^{N}p^n=\frac{p^{N+1}-1}{p-1}$.

5.3 Riemann integration

Definition 1. Partition finite interval $[a,b]$ into n subintervals with end points $a=x_0<x_1<\cdots<x_n=b$, choose $c_k\in[x_{k-1},x_k]$, then $\sum_{k=1}^{n}f(c_k)(x_k-x_{k-1})$ is called the *Riemann sum.* If, as the partition gets finer, for arbitrary choice of c_k, the Riemann sum converges to a number I, then we call I the *Riemann integral* of f from a to b, denoted as $\int_a^b f(x)dx$.

Note that this definition works only for bounded function on finite interval.

When $b<a$ we define $\int_a^b f dx=-\int_b^a f dx$.

As a (somewhat challenging) exercise on the $\epsilon-\delta$ language, one can prove the following:

Theorem 9. *$\int_a^b f(x)dx$ exists if and only if the upper sum and lower sum defined in section 5.1 both converges to the same limit as $n\to\infty$. And $\int_a^b f(x)dx$ equals that limit.*

Geometrically, the definite integral is the "signed area" between the graph of f and the x-axis.

Properties:

(1) $\int_a^b c_1f(x)+c_2g(x)dx=c_1\int_a^b f(x)dx+c_2\int_a^b g(x)dx$.
(2) $\int_a^b f(x)dx+\int_b^c f(x)dx=\int_a^c f(x)dx$.
(3) $f\geq 0$, $a<b$, then $\int_a^b f(x)dx\geq 0$.

Example: Calculate:

(1) $\int_a^b 1dx$
(2) $\int_{-1}^1 f(x)dx$ such that $f(0)=0$ and $f(x)=1$ for $x\neq 0$
(3) $\int_{-1}^1 |x|dx$
(4) $\int_{-1}^1 \sqrt{4-x^2}dx$

(5) $\int_{-1}^{1} \sqrt{4-x^2} - |x| dx$
(6) $\int_0^3 f(x)dx$ where $f(x) = 1-x$ when $0 \le x \le 2$ and $f(x) = x-3$ when $2 < x \le 3$.

5.4 Fundamental Theorem of Calculus

Theorem 10. *If f is continuous on $[a,b]$, then*
(1) $\frac{d}{dt}\int_a^x f(t)dt = f(x)$
(2) If $F' = f$, then $F(b) - F(a) = \int_a^b f(x)dx$

Examples: Calculate:
(1) $\int_1^2 \frac{1}{x}$
(2) $\int f(x)dx$, where $f(x) = \begin{cases} \sin x, & x \le 0 \\ x, & x > 0 \end{cases}$.
(3) $\frac{d}{dx}\int_0^{x^2} e^{-t^2} dt$
(4) $\int_{-1}^{1} 1 + x + e^x dx$

Remark 7. $\int_a^b f(t)dt$ and $\int f(t)dt$ look similar but they are conceptually COMPLETELY DIFFERENT. Fixing f, the former formula depends on a and b, while the second formula is a function of t. The FTC tells us that when f is continuous, they are related by the following formula: $\int f(x)dx = \int_a^x f(t)dt + C$.

Theorem 11. *Any continuous function is integrable.*

Theorem 12. *(Intrgral MVT) If f is continuous, there is some c in $[a,b]$ such that $f(c) = \int_a^b f(t)dt$*

This is a consequence of the intermediate value theorem and positivity. This is weaker than the MVT but sometimes more convenient.

Substitution

From Chain rule $F(g(x))' = g'(x)F'(g(x))$, we know that if the function we are integrating can be written in the form of $g'(F'(g(x)))$ then its indefinite integral is $F(g(x))$. Or, $\int u'(x)f(u(x))dx = \int f(u)du$.

Example: $\int \sin x \cos x^2 dx$

$\int \sin(2x-1)dx$

Review for prelim 3

4.4 Concavity and sketching
Concave up: f' increase.
Concave down: f' decrease.
Inflection point: f' exist or is infinity, and concavity changes.
Second derivative test: $f'(c) = 0$, $f''(c) < 0$ implies local max, $f'(c) = 0$, $f''(c) > 0$ implies local min.
Graphing: domain, symmetry, critical points, inflection points, increasing/decreasing, concave up/concave down, asymptotes, intersection with x- and y-axis.
Example: Sketch $f(x) = x^3 e^{-x}$.

4.5 L'Hospital's Rule
To use L'Hospital's rule, we must make sure (1) the limit is of the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and (2) the derivative of the numerator and denominator has a limit.
Example: $\lim_{x\to 0}(\sin x)^{\tan x}$

4.6 Example: The longest side of a right-angled triangle has length 1. What is its largest possible area?

4.8 Indefinite integral
Remember the constant C when calculating indefinite integral.

Chapter 5:
Example: Find the area between $y = x^3 - x$ and the x axis.
Calculate $\int_1^2 \frac{dx}{\sqrt{5-2x}}$.

5.6 Substitution for definite integral, symmetry, area between curves

Substitution for definite integral: $\int_a^b u'(x) f(u(x))dx = \int_{u(a)}^{u(b)} f(u)du$.

Example: $\int_{\pi/4}^{\pi/3} (\cos x)^{1/3} \sin^3 x dx$.

Symmetry: Example: $\int_{-1}^{1} \frac{\sin x}{|x|} dx$.

Area between curves:

(1) Find the intersection points first. Example: find the area between $y = \sin x$, $y = \cos x$, $x = 0$ and $x = 2\pi$.

(2) Sometimes it is more convenient to integrate in the y direction. Example: $\int_0^1 \sin^{-1} x dx$.

More examples of substitution: $\int \sec x dx$ (Hint: let $u = \sin x$)

$\int \frac{dx}{\sqrt{a^2-x^2}}$ (Hint: let $x = a\sin t$)