DAY I

Activity I: What is the sum of the angles of a triangle? How can you show it? How about a quadrilateral (a shape with 4 sides)? A pentagon (a shape with 5 sides)? Can you find the sum of their angles by cutting them into triangles?

Now imagine yourself walking on the three sides of a triangle, and keep the triangle to your left. How many left turns do you need to make before you are back? What is the sum of all the turning angles? If you do the same on a square, what is the sum of all the turning angles? How about shapes with more sides? How do you explain your answer?

A surface is a space where at any point that is not on the boundary, there is a whole circle worth of possible directions that you could move. For example, the plane is a surface, and the triangle is a surface with boundary. Can you think of any other surfaces?

We can study a surface by measuring lengths and angles (which we call "geometry"), and we can also study it through their "topology", i.e. by imagine them as made from very malleable material that can easily be stretched and deformed (e.g. a triangle has the same topology as a circle, a quadrilateral, or a punctured sphere).

If a surface can be made with small discs that can be "flattened" without stretching and tearing, we call it "flat". Just like before, we can do geometry (draw straight lines, measure lengths and angles) on a flat surface. But first, let's look at some examples of flat surfaces.

Flat cylinder

We can glue the two opposite sides of a rectangle together without twisting. If you walk on a straight line on a flat cylinder, you will either hit the boundary at some time or be back to your starting position, and the second case will never happen on a plane or a triangle. And unlike the plane or the triangle it has two boundaries.

What would you see if you live on a flat torus? What is the relationship between it and a room with 2 mirrors?

Möbius band

If we do a twist before we glue the two opposite sides, what we get is called a Möbius band. Can you describe how the geometry and topology of a Möbius band?

Unlike the flat cylinder, the triangle and the plane, the Möbius band has only one side. We call surfaces with one side "non-orientable" and surfaces with two sides "orientable".

Is the surface still non-orientable if you twist twice before gluing? How about 3 times?

Activity II What would happen if try to color a side of the Möbius strip? What happens if you cut a Möbius band in the middle? How about cut out the boundary cylinder? How do you distinguish a Möbius strip from a triangle or a flat cylinder? (Suggested by Yao Liu)

Activity III Tic-Tac-Toe on a Mobius band. (Suggested by Kristen Pueschel)

Flat torus and Klein bottle

If we glue the two boundaries of the flat cylinder together, keeping the two sides aligned (by allowing folding, or by using your imagination), we get something called flat torus. What are some straight lines on the flat torus? What would you see if you live on a flat torus?

Activity IV Tic-Tac-Toe on a flat torus. (Suggested by Kristen Pueschel)

In both III and IV, the first player has a winning strategy.

If we do the same gluing but flip the sides, what we get is called the Klein bottle.

More flat surfaces

Activity V Make a flat surface by cutting out the shape in your handout and glue the labeled edges. Can you relate it to the flat torus?

Video clip of deformation retract from punctured torus to this flat surface.

There is a surprising connection between the geometry and topology of flat surfaces: two "finite" flat surfaces are topologically the same if and only if they have the same orientability, number of boundaries and total turning angle.

In the orientable case, a flat surface with total turning angle $n \times 360°$ (left turn is positive, right turn is negative) and m boundaries is topologically a surface with $(2 - m - N)/2$ handles and with m discs removed. In the non-orientable case, a flat surface with total turning angle $n \times 360°$ and m boundaries is topologically a surface with $m + (2 - m - N)$ discs removed and $(2 - m - N)$ Möbius bands glued.

By cutting the surface into triangles, and then use induction, we know that:

$$N = \text{number of triangle} + \text{number of points} - \text{number of sides}$$

(Gauss-Bonnet formula) As triangulation of a surface can be preserved by any deformation, the total turning angle should remain the same also.
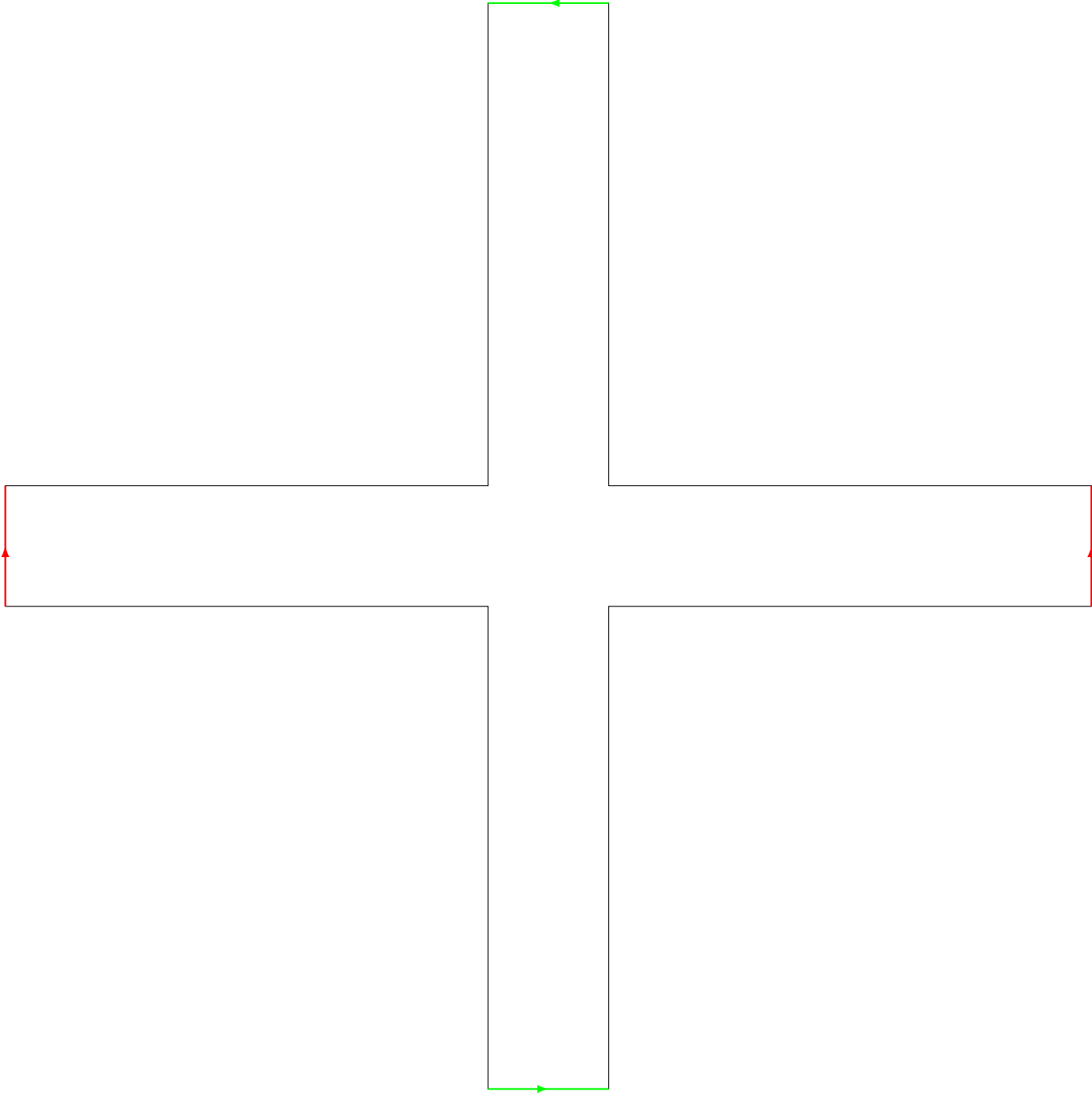
What is the orientability, number of boundaries and total turning angles of the flat surfaces you have seen? Can you build other flat surfaces and do the same calculation?

Not all surfaces can be made flat, e.g. the sphere. We will talk more about them next week.

Handouts starting from next page.
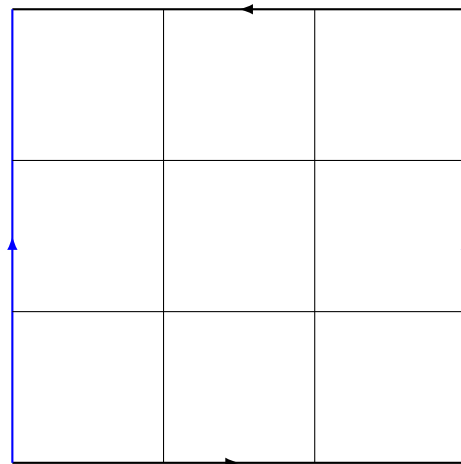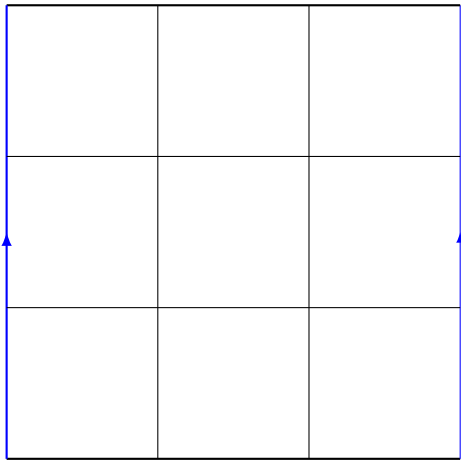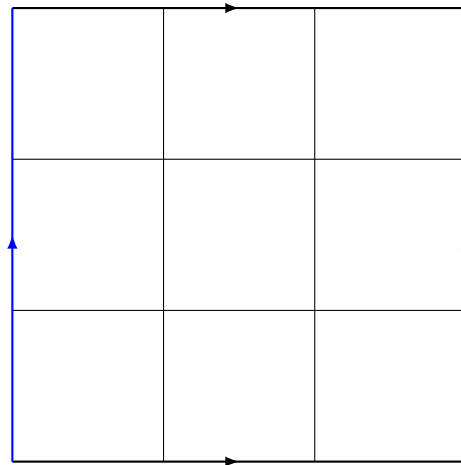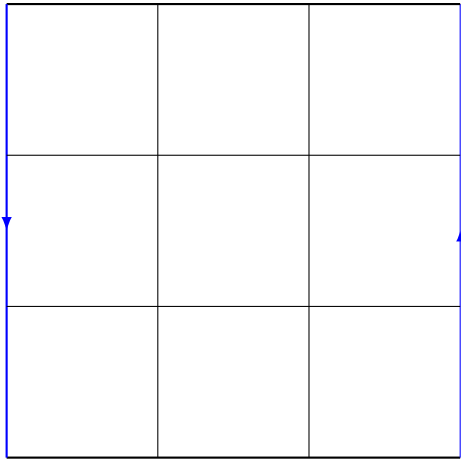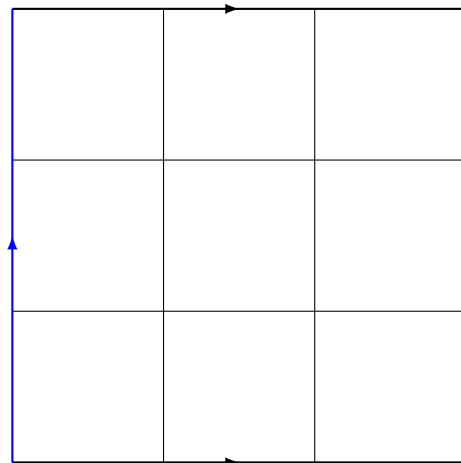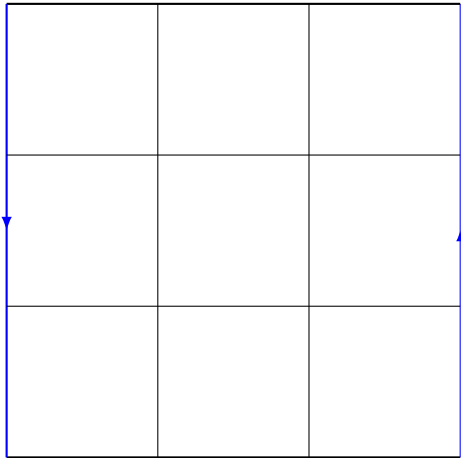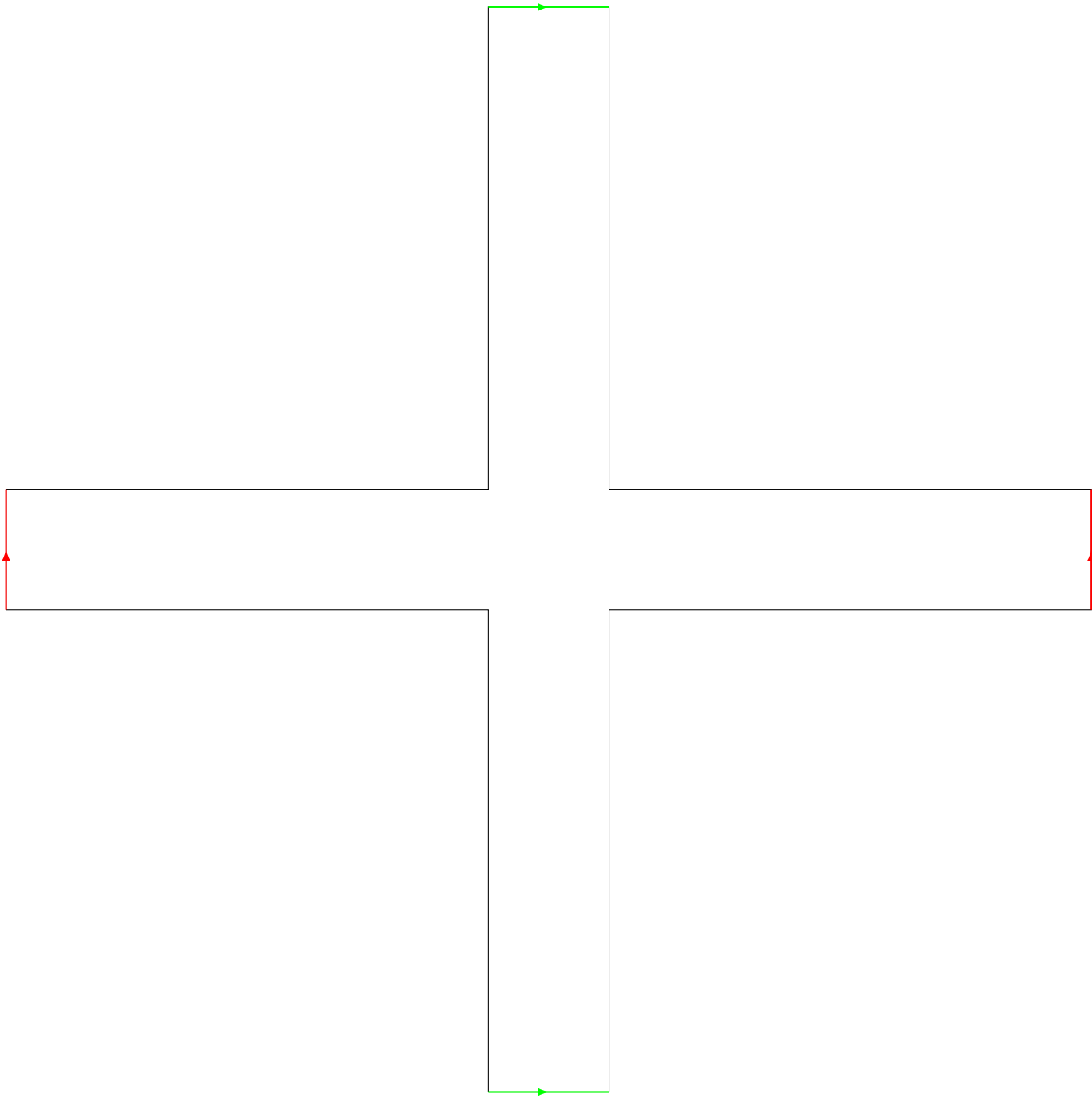
DAY II

Review

Last Saturday we talk about several types of flat surfaces: **the flat cylinder**, **Möbius strip** and **the flat torus**. We introduced the concept of orientability (a surface is orientable if and only if it is two-sided), and how to find the topological type of a flat surface by orientability, number of boundaries and the total turning angle (when orientable, if it has m boundaries and the total turning angle is $N \times 360°$, it has $(2 - m - N)/2$ handles).

Remark: At the very end of last Saturday, we saw an example showing that when there are "cone points", the angle deficit of those points should be counted into turning angles.

Klein Bottle

Klein bottle can be made by gluing the two boundary circles of a flat cylinder in reverse direction.

Show video.

Show paper model for punctured Klein Bottle.

Activity I Why is it the punctured Klein bottle? What happens if we cut it in half? (Suggested by Yao Liu)

Activity II Who will win in a tic-tac-toe on a Klein bottle? (Suggested by Kristen Pueschel)

Real Projective Plane

Another important closed surface is the real projective plane, which can be formed by either gluing a disc to a Möbius strip, or gluing the opposite points of the boundary circle of a disc together. Why do these two procedures produce the same result?

Spherical geometry

A flat surface is "uncurved", a coned surface is "curved only at one point", and the next least complicated example would be a surface that is "curved evenly". Sphere is one such example.

Antivity III What is the shortest path for a bug to travel from a pole of a sphere to an opposite pole? (Suggested by Farbod Shokrieh and John Pike)

As spheres are "uniformly curved", we should guess that the total turning angle when walking along the boundary of a spherical region should be proportional to its area. What is the proportion?

Activity IV What is the relationship between the turning angle of a polygon on a sphere of radius 1 and its area?

It is not possible to "flatten" a sphere without deformation, but there are several standard ways of rendering spherical shapes for map-making and other purposes.

Activity V `https://gmaps-samples.googlecode.com/svn/trunk/poly/puzzledrag.html` (Suggested by Farbod Shokrieh and John Pike)

Spherical tiling and platonic solids

We call a polygon regular if all its sides and angles are the same.

Activity VI Which regular polygons can tile the plane?

As spherical regular k-polygons have larger angles than planar k-gons, only 3-, 4- and 5-gons may tile a sphere. The 5 possible tilings arises from the 5 platonic solids.

What is the total angle deficit of platonic solids?