https://wuchenxi.github.io/math_circle.pdf

We can write a real number in $[0, 1)$ into decimals (in base 10). For example:

$$1/3 = 0.333333...$$

The process goes as follows: let $0 \leq a < 1$, define

$$x_0 = a, n_0 = 0$$

Now for any positive integer k,

$$n_k = \lfloor 10x_{k-1} \rfloor, x_k = 10x_{k-1} - n_k$$

Here $\lfloor b \rfloor$ is the largest integer no more than b. Now we call the **decimal expansion** of $1/3$ as

$$n_0.n_1n_2n_3...$$

Example 1. *To find decimal expansion of* $1/3$*, we have:*

$$x_0 = 1/3, n_0 = 0$$

$$x_1 = 1/3, n_1 = 3$$

$$x_2 = 1/3, n_2 = 3$$

$$\ldots$$

Hence $1/3 = 0.3333...$

If we replace "10" with 2 we have **binary (base 2) expansion** of real numbers.

Example 2. *For* $1/3$*, we now get:*

$$x_0 = 1/3, n_0 = 0$$

$$x_1 = 2/3, n_1 = 0$$

$$x_2 = 1/3, n_2 = 1$$

$$x_3 = 2/3, n_3 = 0$$

$$\ldots$$

Hence $(1/3)_2 = 0.01010101010101..._2$.

We can also replace "10" with something which is not an integer, e.g. $5/3$.

Example 3. *To find* $5/3$ *expansion of* $1/3$*, we have*

$$x_0 = 1/3, n_0 = 0$$

$$x_1 = 5/9, n_1 = 0$$

$$x_2 = 25/27, n_2 = 0$$

$$x_3 = 44/81, n_3 = 1$$

$$x_4 = 220/243, n_4 = 0$$

$$x_5 = 371/729, n_5 = 1$$

$$\ldots$$

So $(1/3)_{5/3} = 0.0010101010001000100 0..._{5/3}$

Question 4. *Write down* $1/2$ *in base* $5/3$*, up to six digits. How about* $2/3$*?*

Answer:

$$(1/2)_{5/3} = 0.01010000100000010.._{5/3}$$

$$(2/3)_{5/3} = 0.10000101000101001010.._{5/3}$$

Question 5. *Can you write down some number in* $[0, 1)$ *where the* $5/3$ *expansion is periodic?*

Answer: There are many. For example, $(9/16)_{5/3} = (0.01010101...)_{5/3}$. The key idea is that for n_k to be periodic, we need to get x_k to be periodic.

Question 6. *Can you show that there can not have three successive* 1 *in the* $5/3$ *expansion of any number in* $[0, 1)$*?*

Answer: A more direct proof is that all x_k are between 0 and 1, hence $n_k = 1$ means $x_{k+1} < 2/3$. If $n_{k+1} = 1$, then $x_{k+1} < 1/9$, hence $n_{k+2} = 0$. Alternatively one can use the answer to Question 9.

Question 7. *What is the relationship between the* $5/3$ *expansion of* $a \in [0, 1)$ *and of* $5a/3 - \lfloor 5a/3 \rfloor$*?*

Answer: The latter is the former "shifted by one" to the left (i.e. deleting the first digit after "."). For example,

$$(1/2)_{5/3} = 0.01010000100000010.._{5/3}$$

$$(5/6)_{5/3} = 0.1010000100000010.._{5/3}$$

Question 8. *Suppose* $0 \leq x < y < 1$*. What is the relationship between the* $5/3$*-expansion of* x *and* y*?*

Answer: The 5/3 expansion of y is larger under **lexicographic order**: if $(y)_{5/3} = 0.n_1n_2\dots$, $(x)_{5/3} = 0.m_1m_2\dots$, suppose k is the smallest integer where $n_k \neq m_k$, then $n_k > m_k$.

Question 9. *Can you describe all the possible sequences of* 0 *and* 1 *that can be the* 5/3*-expansion of some number in* $[0, 1)$*?*

Answer: We can pick numbers in $[0, 1)$ that gets closer and closer to 1 and evaluate their 5/3-expansion, and as these numbers approach 1, there is a "limiting sequence":

$$\lim_{x\to 1^-}(x)_{5/3} = (0.110000101000101001010...)_{5/3}$$

Now Question 8 implies that for any number in $[0, 1)$, the 5/3-expansion is no more than $\lim_{x\to 1^-}(x)_{5/3}$. Combined with Question 7 we get the necessary condition below:

A sequence $0.s_1s_2\dots$ is a 5/3 expansion iff after deleting any prefix, it is always lexicographically smaller than $\lim_{x\to 1^-}(x)_{5/3}$.

This is actually a sufficient criteria as well.

We can replace 5/3 with some other number, e.g. the golden ratio $\phi = \frac{\sqrt{5}+1}{2}$.

Question 10. *What is* $\lim_{x\to 1^-}(x)_\phi$*?*

Answer: It is 0.10101010101010....

As a consequence, the sequences that may be ϕ expansions, must correspond to paths on the directed graph with 2 vertices labeled as 0 and 1, and where there is an arrow from 0 to 0 and 1, and an arrow from 1 to 0.

Reference: `https://en.wikipedia.org/wiki/Non-integer_base_of_numeration`