1. Consider a square grid in $\mathbb{R}^2$, consisting of horizontal and vertical lines of the form $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. Pick a direction with rational slope p/q, define the **cutting sequence** $C(p/q)$ as an infinite sequence with 3 letters a, b, c, constructed as follows:

Starting at $(0,0)$ and travel in the p/q direction, whenever we cross the interior of a horizontal edge write a a, whenever we cross a vertical edge write a b, whenever we pass through a point in $\mathbb{Z} \times \mathbb{Z}$ write a c.

For example,

$$C(3/2) = abacabacabac\ldots$$

2. How do we get from $C(p/q)$ to $C(p/(q+p))$?

Consider the map $(x, y) \mapsto (x-y, y)$, then it sends the ray with slope $p/(p+q)$ to the ray with slope p/q, and the horizontal edges remain the same while vertical edges turned into diagonal lines of the form $x + y \in \mathbb{Z}$. So as a consequence, a should be replaced by ba, b by b, and c by bc.

Hence

$$C(3/5) = babbabcbabbabcbabbabc\ldots$$

3. Can you do the same for $C((p+q)/q)$? How about $C((2p+q)/(p+q))$?

4. Problem 3 tells us how to relate $C(k)$ and $C((2k+1)/(k+1))$. Iterate the map $k \mapsto (2k+1)/(k+1)$, starting at some $k > 0$, what is the limit?

5. Let $k_0 = 3/2$, $k_{n+1} = (2k_n + 1)/(k_n + 1)$, $C_n = C(k_n)$, would the finite prefixes of C_n stabilize?

6. Show that when k is irrational, $C(k)$ is a non-periodic sequence of a and b, and any subsequence of $C(k)$ appear infinitely many times.