Math 561

- Lectures (on Zoom): 11 am-12:15 pm, Tu Th
- Office hours (on Zoom): 1-2 pm TF [1], or by appointment.
- Weekly HW assignments.
- Grades: 10% HW (lowest 2 dropped), 35% Midterm (in class), 55% Final.
- Letter grade range: $[90, 100]$: A, $[75, 90)$: B or AB, $[60, 75)$: BC or C.

[1] Office hours for some weeks may be moved to other time slots.

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- Research area: Ergodic Theory, Geometry Group Theory

History of differential geometry

- Apollonius (3rd century BCE): "curvature" of conic sections.
- Nicole Oresme (14th century): curvature of a curve.
- Christiaan Huygens, Isaac Newton (17th century): methods for calculating curvature of curves.
- Carl Friedrich Gauss (1825, 1827, *General Investigations of Curved Surfaces*): basically what we will learn this semester.
- Bernhard Riemann (1854, *On the Hypotheses Which Lie at the Foundations of Geometry*): curvature in higher dimensions.
- Charles Ehresmann (1905-1979): modern language of differential geometry.

Michael Spivak's *A Comprehensive Introduction to Differential Geometry* Vol. 2 has the English translation of these two texts by Gauss and Riemann.

Topics we will cover for this semester

- Plane curves (about 2 weeks)
- Space curves (about 2 weeks)
- Extrinsic geometry of surfaces in $\mathbb{R}^3$ (about 3 weeks)
- Intrinsic geometry of surfaces in $\mathbb{R}^3$ (about 3 weeks)
- Gauss-Bonnet theorem (about 3 weeks)
- Textbook: do Carmo, Differential Geometry of Curves and Surfaces

General Advices

- Make sure you understand the definition of every concept clearly before moving on.
- Use "geometric intuition" to guide your calculation.

Some key ideas in differential geometry

- **Coordinate charts**: When doing computation, we often describe curves and surfaces, at least locally, as image of maps. However we often want our geometric quantity to be invariant under change of coordinate.
- **Distance**: The distance between two points of an object can be measured by the length of the shortest path within the object. If a map from a curve or surface to another preserves distance we call it an **isometry**, and quantities preserved by isometries are called **intrinsic**.
- **Parallel transport, or connection**: We want a way to relate vectors at one point of the object with those at another.
- **Curvature**: From parallel transport we can get varies quantities that measures how much an object deviates from being "flat". They can be either intrinsic, or extrinsic.

2 Plane Curves

Basic Definitions

Definition 2.1.1

A (parameterized) **plane curve** is a continuous function from an interval to $\mathbb{R}^2$. In other words, a map $\gamma : I \to \mathbb{R}^2$, such that $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, γ_i are both continuous.

Definition 2.1.2

A plane curve is said to be differentiable, C^k, or smooth if the function is differentiable, C^k, or smooth.

Example 2.1.3

Let $\gamma_k : \mathbb{R} \to \mathbb{R}^2$ be $\gamma_k(t) = (t, t^k|t|)$. Then γ_0 is a plane curve that is not differentiable, γ_k is C^k but not C^{k+1}.

Definition 2.1.4

A differentiable plane curve is called **regular** if $d\gamma_1/dt \neq 0$ or $d\gamma_2/dt \neq 0$ at any point in I.

Definition 2.1.5

A plane curve is called **simple** if it is injective. For simple curves sometimes we identify the curve with its image.

Example 2.1.6

The smooth curve $\alpha : \mathbb{R} \to \mathbb{R}^2$, $\alpha(t) = (t^2, t^3)$ is simple but not regular.
The smooth curve $\beta : \mathbb{R} \to \mathbb{R}^2$, $\beta(t) = (t^2 + t, t^3 + t^2)$ is regular but not simple.

Definition 2.1.7

Let $\alpha : I \to \mathbb{R}^2$, $\beta : I' \to \mathbb{R}^2$ be two plane curves. If there is a continuous (or differentiable, or C^k, or smooth) bijection $i : I \to I'$, whose inverse is also continuous (or differentiable, or C^k, or smooth), such that $\beta = \alpha \circ i$, then we call α and β are continuous (or differentiable, or C^k, or smooth) **reparameterization** of one another. If $i' > 0$ this reparameterization is **orientation preserving**, otherwise it is **orientation reversing**.

Example 2.1.8

Let $I = (0, 2\pi)$, $\alpha(t) = (\cos(t), \sin(t))$. The curve is smooth, regular and simple. Let $I' = \mathbb{R}$, $\beta(t) = ((t^2 - 1)/(t^2 + 1), 2t/(t^2 + 1))$. The curve is also smooth, regular and simple, and the orientation reversing smooth reparametrization between I and I' is:

$$I \to I' : t \mapsto -\tan((t - \pi)/2)$$

$$I' \to I : t \mapsto 2\arctan(-t) + \pi$$

- The curve in the example is the unit circle with $(1, 0)$ removed. The first parameterization is by the angle between vector $(1, 0)$ and (x, y). The second parameterization is by the intersection of the y axis and the line passing through $(1, 0)$ and (x, y).
- It is easy to see that under reparameterization, the image of the curve remain unchanged.

Arc Length and Arc Length Parameterization

Recall that in $\mathbb{R}^2$, the Euclidean norm is $\|x\| = (x \cdot x)^{1/2} = \sqrt{x_1^2 + x_2^2}$, the Euclidean distance between x and y is $\|x - y\|$.

Definition 2.2.1

Let $\gamma = (\gamma_1, \gamma_2)$ be a regular C^1 curve. The **length** of γ is defined as $\int_I \|\gamma'\| dt$

Definition 2.2.2

The **distance** between $\gamma(a)$ and $\gamma(b)$ along the curve is defined as $|\int_a^b \|\gamma'\| dt|$

Definition 2.2.3

If γ is a C^1 regular curve, $t_0 \in I$, then the function $s(t) = \int_{t_0}^t \|\gamma'(u)\| du$ is called the **arc length**. The distance from $\gamma(a)$ to $\gamma(b)$ along the curve is $|s(a) - s(b)|$.

Justification of this definition of length

Theorem 2.2.4

If γ is a C^1 regular curve, parameterized by a finite interval $I = (0, 1)$, then

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \|\gamma(i/n) - \gamma((i+1)/n)\| = \int_0^1 \|\gamma'\| dt$$

Proof idea

Mean value theorem implies that

$$\gamma(i/n) - \gamma((i+1)/n) = (\gamma_1'(s_1)/n, \gamma_2'(s_2)/n)$$

For some $s_1, s_2 \in [i/n, (i+1)/n]$. Now apply the uniform continuity of γ'.

The function s is a invertible function as it is C^1 with positive derivative.

Definition 2.2.5

The **arc length parameterization** of a C^1 regular curve γ is defined as $\beta = \gamma \circ s^{-1}$.

- The arc length of an arc length parameterization is $t \mapsto t - t_0$.
- Just like the arc length function depends on the choice of t_0, a different choice of t_0 will result in a precomposition by some map $s \mapsto s + C$.

Review for the last lecture

- C^k Plane curves
- Regularity, simplicity
- Reparameterization, orientation
- Length of a plane curve, distance between two points along the curve
- Arc length function
- Arc length parameterization

True or false: If α is a C^1 regular plane curve, $\gamma = \alpha \circ s^{-1}$ the arc length parameterization. Then γ is an orientation preserving reparameterization of α.

Remark

The textbook starts with space curves. So this first chapter is just an extended introductory example, we will move on to textbook materials next week!

Example 2.2.6

Consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $t \to ((t^2-1)/(t^2+1), 2t/(t^2+1))$.

- Let $t_0 = 0$, then the arc length is

$$s(t) = \int_0^t \sqrt{(\frac{2t(t^2+1)-(t^2-1)\cdot 2t}{(t^2+1)^2})^2 + (\frac{2(t^2+1)-4t^2}{(t^2+1)^2})^2} dt$$

$$= \int_0^t \frac{2}{t^2+1} dt = 2\arctan(t)$$

- The total length of γ is $s(\infty) - s(-\infty) = 2\pi$.
- The arc length parameterization is

$$(\frac{\tan^2 \frac{s}{2} - 1}{1 + \tan^2 \frac{s}{2}}, \frac{2\tan\frac{s}{2}}{\tan^2\frac{s}{2} + 1}) = (-\cos(s), \sin(s))$$

where $I = (-\pi, \pi)$.

Calculus digression: Chain Rule

Theorem 2.2.7

If $F:\mathbb{R}^m \to \mathbb{R}^n$, $G:\mathbb{R}^n \to \mathbb{R}^r$ are both C^1 functions, then the **derivative**, *or* **Jacobian matrix**, *of F at $x \in \mathbb{R}^m$ is defined as $JF(x) = [\partial_j F_i|_x]_{n\times m}$, where $F = (F_1, \ldots, F_n)$. The chain rule for the derivative of $G \circ F$ at $x \in \mathbb{R}^m$ is*

$$J(G \circ F)(x) = JG(F(x))JF(x)$$

Example 2.2.8

If F is the polar coordinate change $(r,\theta) \to (r\cos\theta, r\sin\theta)$, G is $(x,y) \mapsto x^2+y^2$, $\phi = G\circ F$, then

$$\begin{bmatrix} \partial_r\phi & \partial_\theta\phi \end{bmatrix} = \begin{bmatrix} 2r\cos\theta & 2r\sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = \begin{bmatrix} 2r & 0 \end{bmatrix}$$

Arc length and reparameterization

Theorem 2.2.9

Let α and β be two C^1 regular parameterizations of the same curve, $\beta = \alpha \circ i$, s_α and s_β are the arc length functions, then $s_\beta = \pm(s_\alpha \circ i + C)$ for some constant C.

Proof.

$$s_\beta(t) = \int_{t_0}^{t} \|\beta'(u)\| du = \int_{t_0}^{t} \|i'(u)\alpha'(i(u))\| du$$

$$= sign(i') \int_{t_0}^{t} \|\alpha'(i(u))\| i'(u) du = sign(i') \int_{i(t_0)}^{i(t)} \|\alpha'(v)\| dv$$

$$= sign(i')(s_\alpha \circ i + C)$$

□

As a result, we have:

Theorem 2.2.10

Distance between two points on a curve is unchanged by reparamterization.

Theorem 2.2.11

Let α and β be two C^1 regular parameterizations, $\beta = \alpha \circ i$, i is invertible and C^1. Then the arc length parameterization of β and α differs by a precomposition of $x \mapsto sign(i')x - C$ for some constant C.

Proof.

Let e be $x \mapsto sign(i')x - C$, then

$$\beta \circ s_\beta^{-1} = \beta \circ i^{-1} s_\alpha^{-1} \circ e = \alpha \circ s_\alpha^{-1} \circ e$$

□

True or false: If α and β are reparameterizations of one another, then the arc length function of α and β differ by a constant.
Answer: False. Need to precompose with i, and possibly multiply by -1 depending on orientation.

Curvature

For now assume all plane curves involved to be simple.

Definition 2.3.1

Let γ be a C^2-regular plane curve, suppose γ is an arc length parameterization. Then the **curvature** at $\gamma(t_0)$ is defined as $\|\gamma''(t_0)\|$. The **signed curvature** is $\|\gamma''(t_0)\|$ if $\gamma''(t_0)$ is in the direction of γ' rotated counterclockwise by $\pi/2$ (i.e. on the left hand side of γ'), $-\|\gamma''(t_0)\|$ if otherwise.

Theorem 2.2.11 implies that the curvature is invariant under reparameterization. The signed curvature is invariant under orientation preserving reparameterization.

Geometric meaning of curvature

If γ is arc length parameterization, then $\|\gamma'\| = 1$, so $\gamma'(t)$ is a **unit tangent vector** of the curve at $\gamma(t)$. Hence, curvature measures **how fast the unit tangent vector turn when one moves along the curve at unit speed**.

Alternatively, we can also call γ' turned counterclockwise by $\pi/2$ the **unit normal vector**. In this case, we can say curvature measures **how fast the unit normal vector turn when one moves along the curve at unit speed**.

Curvature under general regular parameterization

If γ is not an arc length parameterization, we shall use chain rule to calculate the curvature of γ at $t_0 \in I$.
Let γ_1 be the arc length parameterization obtained via γ, which is $\gamma_1 = \gamma \circ s^{-1}$, $s(t) = \int_{t_0}^{t} \|\gamma'\| ds$. Then by chain rule and quotient rule:

$$\gamma_1'(t) = \frac{\gamma' \circ s^{-1}(t)}{\|\gamma' \circ s^{-1}(t)\|}$$

$$\gamma_1'' = \frac{\frac{\gamma'' \circ s^{-1}}{\|\gamma' \circ s^{-1}\|}\|\gamma' \circ s^{-1}\| - (\frac{\gamma'' \circ s^{-1}}{\|\gamma' \circ s^{-1}\|} \cdot \frac{\gamma' \circ s^{-1}}{\|\gamma' \circ s^{-1}\|})\gamma' \circ s^{-1}}{\|\gamma' \circ s^{-1}\|^2}$$

Now set $t = 0$, we get the curvature

$$\frac{\| \|\gamma'(t_0)\|^2 \gamma''(t_0) - (\gamma'(t_0) \cdot \gamma''(t_0))\gamma'(t_0)\|}{\|\gamma'(t_0)\|^4}$$

$$= \frac{\text{Area of parallelogram formed by } \gamma' \text{ and } \gamma''}{\|\gamma'\|^3}$$

A consequence of the formula above is that if two regular C^2 curves, up to reparameterization, have the same derivatives and 2nd order derivatives at a point (called **osculating**), they must have the same radius at the point also.

Example 2.3.2

The circle with a point removed:

$$I = (0, 2r\pi), \gamma(t) = (r\cos(t/r), r\sin(t/r))$$

It is easy to see that this is an arc length parameterization, the curvature at $\gamma(t)$ is

$$\|\gamma''(t)\| = \|(-\cos(t/r)/r, -\sin(t/r)/r)\| = 1/r$$

Because $(-\cos(t/r), -\sin(t/r))$ is γ' turned counterclockwise by $\pi/2$, the signed curvature is also $1/r$.

This example, together with the discussion on the previous page, implies the earliest definition of curvature of a curve: **The curvature of a curve is the reciprocal of the radius of the osculating circle of the curve.**

Example 2.3.3

Consider

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \gamma(t) = (t, t^2)$$

The curvature at $t = 0$ is

$$\frac{\| \|(1,0)\|^2 (0,2) - ((1,0)\cdot(0,2))(1,0)\|}{\|(1,0)\|^4} = 2$$

It is also the signed curvature.

Fundamental Theorem of the Local Theory of Plane Curves

Theorem 2.3.4

Given any interval I containing zero, any smooth real valued function k on I, there is a regular smooth plane curve $\gamma : I \to \mathbb{R}^2$, such that γ is an arc length parameterization, and γ'' is k times γ' turned counterclockwise by $\pi/2$. Furthermore, any two such plane curves are identical up to rigid motion.

In other words, up to a rigid motion, the signed curvature function k **completely characterized the shape of a plane curve**.

The proof requires Picard's theorem on ODEs:

Theorem 2.3.5

Let I be any interval, $t_0 \in I$, $x_0 \in \mathbb{R}^n$, F is a function from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ that is continuous and Lipschitz on the second parameter (i.e. there is some number L such that $\|F(t,x) - F(t,y)\| \leq L\|x-y\|$, then the initial value problem

$$y(t_0) = x_0, y' = F(t,y)$$

has a unique solution. Furthermore, if F is smooth, or real analytic, so is the solution.

The proof can be found in textbooks on ordinary differential equations.

Proof of the Fundamental Theorem of the LTPC

Proof.

For existence, apply Picard's theorem to the following IVP:

$$\gamma_1(0) = 0, \gamma_2(0) = 0, v_1(0) = 1, v_2(0) = 0$$

$$\gamma_1' = v_1, \gamma_2' = v_2, v_1' = -kv_2, v_2' = kv_1$$

Proof of the Fundamental Theorem of the LTPC

Proof.

Now we verify that $\gamma = (\gamma_1, \gamma_2)$ satisfies the requirements in the theorem:

$$\|\gamma'(0)\| = \|(1,0)\| = 1$$

$$(\|\gamma'\|^2)' = (\gamma' \cdot \gamma')' = 2\gamma' \cdot \gamma'' = 2(v_1, v_2) \cdot (-kv_2, kv_1) = 0$$

So $\|\gamma'\| = 1$, the arc length function when $t_0 = 0$ would be identity, which implies that γ' is an arc length parameterization.

Proof of the Fundamental Theorem of the LTPC

Proof.

Next, we have

$$\gamma'' = k(-v_2, v_1)$$

Here $(-v_2, v_1)$ is the unit tangent vector (v_1, v_2) rotated counterclockwise by $\pi/2$.

Proof of the Fundamental Theorem of the LTPC

Proof.

For uniqueness, suppose β is the arc length parameterization of another curve that satisfy the assumptions in the theorem. Do a translation to move $\beta(0)$ to $(0, 0)$, and a rotation around $(0, 0)$ to move $\beta'(0)$ to $(1, 0)$, now following the same calculation as before, we know that $\beta_1, \beta_2, \beta_1', \beta_2'$ satisfies the same IVP of $\gamma_1, \gamma_2, v_1, v_2$. Hence by Picard's theorem $\beta = \gamma$. □

Local Canonical Form

Solving the ODE in the prove of the Fundamental Theorem, with $t_0 = 0$, we get the **local canonical form**:

$$x(t) = t - \frac{k_0^2 t^3}{6} + O(t^4), y(t) = \frac{k_0}{2} t^2 + \frac{k_0' t^3}{6} + O(t^4)$$

Here $k_0 = k(0), k_0' = k'(0)$ Any smooth regular plane curve, after rigid motion and arc-length reparameterization, can be turned into the local canonical form.

Example

Example 2.3.6

$I = [-1, 1]$, $k(t) = t$. Then the ODE for $\gamma(t)' = (v_1(t), v_2(t))$ becomes

$$v_1' = -tv_2, v_2' = tv_1, v_1(0) = 1, v_2(0) = 0$$

Let $v_1 = \cos(\theta(t))$, $v_2 = \sin(\theta(t))$, we get $\theta' = t$, $\theta(0) = 0$, so $\theta(t) = t^2/2$. Hence the plane curve becomes

$$t \mapsto (\int_0^t \cos(t^2/2)dt, \int_0^t \sin(t^2/2)dt)$$

Taylor series expansion at 0 up to degree 3 gets the LCF:

$$t \mapsto (t + O(t^4), \frac{t^3}{6} + O(t^4))$$

- There will be no proofs in the exam, but **by the end of the semester, understanding of proofs is needed to fully understand the lectures and get an A in the course**.
- Please come to office hours more if you like geometry but is not yet comfortable with proofs.
- Review analysis and linear algebra.

Review of set-theoretic notations

- A is a set, we call B a **subset** of A, if any member of B is a member of A.
- Subsets can be represented via the notation of **specification**: $B = \{a \in A : \phi(a)\}$. Here ϕ is a sentence depending on a variable a, which can either be true or false (in logic we call it a **predicate**, and B consists of all elements of A that makes ϕ true.
- The **product** of two sets A and B, denoted as $A \times B$, is the set consisting of **ordered pairs** of elements in A and B.
- The **power set** of A, denoted as 2^A, is the set consisting of subsets of A.

- A **map**, or **function**, from a set A to a set B is denoted as $f : A \to B$. We often specify a map by describing how it may send an element of A to an element of B, for example $f : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t, t^2)$. A is called the **domain** and B is called the **codomain**. The set $\{x \in B : \text{there is some } a \in A, x = f(a)\}$ is called the **range**.
- If A is a set, we call $id : A \to A$, $a \mapsto a$, the **identity** map.
- We say two maps are the same iff (1) they have the same domain and codomain; (2) they gives the same value when evaluating on every element of the domain.
- Let A, B and C be three sets, $f : A \to B$, $g : B \to C$ are two maps, then the **composition** $g \circ f$ is defined as $(g \circ f)(a) = g(f(a))$.
- If $f : A \to B$ and $g : B \to A$ satisfies $g \circ f = id$, $f \circ g = id$, we call g the **inverse** of f. A map with an inverse is called a **bijection**.

- An **equivalent relation** on a set A is a subset R of $A \times A$, such that (1) $(a, a) \in R$ for all $a \in A$; (2) $(a, b) \in R$ implies $(b, a) \in R$; (3) $(a, b) \in R$, $(b, c) \in R$ implies $(a, c) \in R$. We sometimes denote $(a, b) \in R$ by aRb.
- An **equivalence class** of A under relation $\sim$ is a subset of the form $\{b \in A : a \sim b\}$ for some $a \in A$. This equivalence class is denoted by $[a]$.
- The set consisting of all equivalence classes of A under $\sim$ is called a **quotient**, denoted as $A/\sim$.

Variable scopes

In mathematical writing, just like in programming, every variable has its scope.

Example 2.3.7

Because $f : x \mapsto \sin x$ and $g : y \mapsto y^2$ are both differentiable, we have

$$\int_0^t f(u)g'(u)du = f(t)g(t) - f(0)g(0) - \int_0^t f'(u)g(u)du$$

- The scope of f and g are in the whole sentence.
- The scope of x and y are within the definition of f and g.
- The scope of t is within the equation.
- The scope of u is within the integration expression.

Variables being renamed within its scope will not change the meaning of a sentence.

Review: Local Theory of Plane Curves

- Definition: A **smooth, regular plane curve** is a smooth function α from interval I to $\mathbb{R}^2$, such that $\alpha' \neq 0$ everywhere on I.
- Properties of a smooth regular plane curve
  - The **arc length function** is a function s defined on I, of the form $t \mapsto \int_{t_0}^{t} \|\alpha'(u)\| du$.
  - The **arc length parameterization** is defined as $\alpha \circ s^{-1}$.
  - Let α_1 be the arc length parameterization, the **signed curvature** at point $\alpha_1(t)$ is a number k, such that α_1'' is k times α_1' turned left by $\pi/2$. $|k|$ is called the curvature. Geometrically, curvature is **the reciprocal of radius of osculating circle**, and **the speed unit tangent vector or unit normal vector rotate when traveling along the curve at unit speed**.

- Fundamental Theorem: a smooth regular plane curve is defined by the signed curvature function under arc length parameterization.
- Reparameterization.

$$\begin{array}{ccc} I_1 & \xrightarrow{\alpha} & \mathbb{R}^2 \\ \downarrow{i} & \nearrow_{\beta} & \\ I_2 & & \end{array}$$

Review Example

Example 2.3.8

Let $\alpha : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t, \sin(t))$.

- Find the arc length function, using $t_0 = 0$.
- Let β be the arc length parameterization. Find $\beta'(0)$, $\beta''(0)$ using chain rule.
- Calculate the signed curvature at $\alpha(0) = (0, 0)$.

- $s(t) = \int_0^t \sqrt{1+\cos^2(r)}dr$.
- $\beta = \alpha \circ s^{-1}$, so

$$\beta'(0) = \frac{\alpha' \circ s^{-1}}{s' \circ s^{-1}}(0) = \frac{1}{\sqrt{2}}(1,1)$$

$$\beta''(0) = \frac{\frac{\alpha'' \circ s^{-1}}{s' \circ s^{-1}} \cdot s' \circ s^{-1} - \alpha' \circ s^{-1} \cdot \frac{(s'' \circ s^{-1})}{s' \circ s^{-1}}}{(s' \circ s^{-1})^2}(0)$$
$$= 0$$

- Calculate the signed curvature at $\alpha(0) = (0,0)$ is 0. This is an inflection point.

Example 2.3.9

Let α is a plane curve defined on $\mathbb{R}$ via arc length parameterization, $k(t)$ is the signed curvature at $\alpha(t)$. We shall write down a formula of α using the proof of fundamental theorem of local theory of plane curves:

$$\left(\int_0^t \cos\left(\int_0^s k(r)dr\right)ds, \int_0^t \sin\left(\int_0^s k(r)dr\right)ds\right)$$

- Recall from the proof, we can solve for α via the system of differential equations:

$$\alpha_1(0) = \alpha_2(0) = 0, v_1(0) = 1, v_2(0) = 0$$

$$\alpha_1' = v_1, \alpha_2' = v_2, v_1' = -kv_2, v_2' = kv_1$$

- The two equations regarding v_1 and v_2 implies that

$$(v_1^2 + v_2^2)' = -2kv_1v_2 + 2v_1v_2 = 0$$

So $v_1^2 + v_2^2 = 1$, we can let $v_1 = \cos(\theta), v_2 = \sin(\theta)$.

- Now we get $\theta(0) = 0$, $\theta' = k$, hence $\theta(t) = \int_0^t k(s)ds$.
- The formula for α can then be obtained via integration.

Rotation index

Theorem 2.4.1

If $\alpha : \mathbb{R} \to \mathbb{R}^2$ is a regular smooth plane curve with period T, $k(t)$ be the signed curvature at $\alpha(t)$, then $\int_0^T k(t)\|\alpha'(t)\|dt = 2n\pi$ for some $n \in \mathbb{Z}$.

Intuitively, if you walk around on a plane and get back to the starting point and starting direction, the total turning you have done must be a multiple of 2π. This n is sometimes called the **rotation index**.

Proof.

Let α_1 be the arc length parameterization of α, where $t_0 = 0$, which is of period $\int_0^T \|\alpha'\| dt = s(T)$. Now suppose $\alpha_1'(u) = (\cos(\theta(u)), \sin(\theta(u)))$, then $\theta'(u) = k(s^{-1}(u))$, hence

$$\int_0^T k(t)\|\alpha'\| dt = \int_0^{s(T)} k(s^{-1}(u)) s'(s^{-1}(u)) ds^{-1}(u)$$

$$= \int_0^{s(T)} \theta'(u) du = 2n\pi$$

□

This n is ± 1 if α is injective on $[0, T)$. This is a consequence of the Gauss-Bonnet theorem which we will discuss towards the end of the semester.

Digression: Discrete Differential Geometry

We can do the above to **regular piecewise linear plane curves** instead of **smooth regular plane curves**.

- Signed curvature becomes signed turning angle.
- Fundamental Theorem becomes: a polygonal curve on the plane is, up to rigid motion, determined by the edge lengths and size of turning angles.
- Gauss-Bonnet becomes: the sum of exterior angles of any polygon is 2π.

1-Manifolds

This is just a preview of what will happen in the second half of the semester. Don't worry if you find it super confusing!

It is natural to try and view curves related via reparameterization as the same geometrical object. Now, for a plane curve $\alpha : I \to \mathbb{R}^2$, I is an open interval, say we want to define the signed curvature function k. However:

- If k is defined on the domain I, then it changes when one change the parameterization (recall the T/F question last Thursday).
- If k is defined on the image $\alpha(I)$, it works well for simple curves, but not non simple curves.

How do we fix this problem?

What if we pick and fix a parameterization α, and any other reparameterization β can be written as $\beta = \alpha \circ i$. Now k is defined on the domain of α, I. However, the problem is that one particular parameterization, α will become something "special". To get back the "symmetry", let's remember all the smooth bijections with smooth inverse $i : I' \to I$, and then FORGET that I is an interval. In other words, we can create an abstract mathematical object, $I^* = (I, \mathcal{C})$, where $\mathcal{C} = \{i : I' \to I\}$, such that:

- For any two maps i and i' in $\mathcal{C}$, $i^{-1}i'$ is smooth with smooth inverse.
- A smooth function on I^* is a function f on I, such that for every i, $f \circ i$ is smooth.

For more complex objects, we may only require elements of $\mathcal{C}$ be injections and not bijections, and we may want a few extra topological constraints. That would become the concept of a **manifold**.

Historical Remarks

The concept of manifold starts with Bernhard Riemann. The word "manifold" goes back further, e.g.

By *synthesis*, in its most general sense, I understand the act of putting different representations together, and of grasping what is manifold in them in one act of knowledge. –Immanuel Kant, *Critique of Pure Reason*

3 Space Curves
- Definitions, arc length, curvature
- Torsion
- Local Theory of Space Curves

Basic Definitions

Definition 3.1.1

A (parameterized) **space curve** is a continuous function from an interval to $\mathbb{R}^3$. In other words, a map $\gamma : I \to \mathbb{R}^3$, such that $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$, γ_i are both continuous.

Concepts like **differentiability**, C^k, **smoothness**, **regularity**, **simplicity**, **reparameterization** are all analogous. We will now focus on **smooth regular space curves**.

Arc Length Function

Definition 3.1.2

Let $\alpha : I \to \mathbb{R}^3$ be a smooth, regular space curve. The **arc length function** is $s(t) = \int_{t_0}^{t} \|\alpha'(r)\| dr$.

The following follows from **fundamental theorem of calculus**.

Theorem 3.1.3

The derivative of the arc length function is $\|\alpha'\|$, hence it is always positive, which implies that the arc length function is a smooth bijection with smooth inverse.

Arc Length Parameterization, Curvature

Definition 3.1.4

Let $\alpha : I \to \mathbb{R}^3$ be a smooth regular space curve, s an arc length function. The **arc length parameterization** is $\alpha \circ s^{-1}$.

Recall that $(f \circ g)(x) = f(g(x))$.

Theorem 3.1.5

If $\beta = \alpha \circ s^{-1}$ is an arc length parameterization of α. Then

1. $\|\beta'\| = 1$.
2. β'' *is orthogonal to* β'.

The length of β'' is called the **curvature**.

Proof.

1. $\|\beta'\| = \|\frac{\alpha' \circ s^{-1}}{\|\alpha' \circ s^{-1}\|}\| = 1.$
2. $\beta'' \cdot \beta' = \frac{1}{2}(\beta' \cdot \beta')' = \frac{1}{2}1' = 0.$

□

Example 3.1.6

Consider $\alpha : \mathbb{R} \to \mathbb{R}^3$, $x \mapsto (x, x^2, x^3)$.

1. α is smooth, regular and simple.
2. Find arc length function and arc length parameterization β.
3. Find β' and β'' at 0.

Orientation

Definition 3.2.1

Let $x_1, \ldots x_n$ be n linearly independent vectors in $\mathbb{R}^n$. We say $\{x_i\}$ form a basis of positive orientation iff $\det([x_1, \ldots, x_n]) > 0$. Otherwise they form a basis of negative orientation. A linear bijection from $\mathbb{R}^n$ to itself is orientation preserving iff it sends standard basis $\{e_1, \ldots, e_n\}$ to a basis of positive orientation, i.e. iff the matrix representation has positive determinant.

Example 3.2.2

- When $n = 1$, T is orientation preserving iff $T' > 0$.
- When $n = 2$, rotation is orientation preserving, while reflection isn't.

Definition 3.2.3

The **cross product** in $\mathbb{R}^3$ is defined as
$(a, b, c) \times (d, e, f) = (bf - ec, cd - af, ae - bd)$.

From multivariable calculus, we know that:

Theorem 3.2.4

- *If a and b are linearly dependent, then $a \times b = 0$.*
- *If a and b are linearly independent, $a \times b$ is orthogonal to both a and b, the length of $a \times b$ is the area of the parallelgram spanned by a and b, and $\{a, b, a \times b\}$ form a basis of $\mathbb{R}^3$ with positive orientation.*

Review for last lecture

- Smooth regular space curve.
- Arc length function.
- Arc length parameterization.
- Properties of arc length parameterization: $\|\alpha'\| = 1$, $\alpha' \cdot \alpha'' = 0$
- Curvature.

- Cross product: (in textbook the symbol is $\wedge$)

$$(x_1, x_2, x_3) \times (y_1, y_2, y_3) = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

- Geometric meaning: $x \times y$ is orthogonal to x and y, of length equals to the parallelogram spanned by x and y, and $x, y, x \times y$ has positive orientation.
- Calculus regarding cross product: $\times$ is **bilinear**, i.e. $(ka + k'b) \times c = ka \times c + k'b \times c$, $a \times (kb + k'c) = ka \times b + k'a \times c$. Hence Leibniz's law for derivatives works.

In other words,

$$(a(t) \times b(t))' = \lim_{h \to 0} \frac{a(t+h) \times b(t+h) - a(t) \times b(t)}{h}$$

$$= \lim_{h \to 0} \frac{(a(t) + ha'(t) + o(h)) \times (b(t) + hb'(t) + o(h)) - a(t) \times b(t)}{h}$$

$$= \lim_{h \to 0} a'(t) \times b(t) + a(t) \times b'(t) + \frac{o(h)}{h}$$

$$= a'(t) \times b(t) + a(t) \times b'(t)$$

Frenet-Serret frame

Let α be a smooth regular space curve with arc length parameterization. At a point $\alpha(s)$ **where the curvature is non zero**, we can define three orthogonal vectors:

- Unit tangent vector: $t = \alpha'(s)$
- Unit normal vector: $n = \alpha''(s)/\|\alpha''(s)\|$
- Unit binormal vector: $b = t \times n$

The three vectors form an orthonormal basis of $\mathbb{R}^3$.
In a neighborhood I_1 of s, where α'' is non zero, we can define the three vectors $t(s)$, $n(s)$ and $b(s)$ as above. This family of **parameterized frames** is called the **Frenet-Serret frames**.

Frenet formulas

Under arc length parameterization, the Frenet-Serret frames change according to a sequence of linear differential equations called Frenet formulas

Remark

Recall that for plane curves, under arc length parameterization, if $t(s) = \alpha'(s)$ is the unit tangent vector, $n(s)$ is $t(s)$ rotated counterclockwise by $\pi/2$, then $t' = kn$, $n' = -kt$, where k is the signed curvature.

Derivation of the Frenet formulas

Recall that the curvature at $\alpha(s)$, denoted as $k(s)$ (**this is NOT the k for plane curves!**), is $\|\alpha''(s)\|$. Hence,

$$t'(s) = \alpha''(s) = \|\alpha''(s)\| \cdot \frac{\alpha''(s)}{\|\alpha''(s)\|} = k(s)n(s)$$

$$\begin{aligned} b'(s) &= t'(s) \times n(s) + t(s) \times n'(s) \\ &= k(s)n(s) \times n(s) + t(s) \times n'(s) \\ &= t(s) \times n'(s) \end{aligned}$$

Hence $b'(s) \perp t(s)$.
Furthermore,

$$b'(s) \cdot b(s) = \frac{1}{2}(\|b(s)\|^2)' = 0$$

So $b'(s)$ is orthogonal to $b(s)$ also.
Hence $b'(s)$ is parallel to $n(s)$, we denote their ratio $\tau(s)$, the **torsion** at point $\alpha(s)$.
Now $b'(s) = \tau(s)n(s)$.

Remark

The plane spanned by t and n is called the osculating plane, so torsion represents **how fast osculating plane rotates under arc length parameterization**.

Cross product and orthonormal basis

If e_1, e_2, e_3 form an orthonormal basis of $\mathbb{R}^3$ of positive orientation, then

$$e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2$$

$$e_2 \times e_1 = -e_3, e_3 \times e_2 = -e_1, e_1 \times e_3 = -e_2$$

$n(s) = b(s) \times t(s)$, hence

$$n'(s) = \tau(s)n(s) \times t(s) + b(s) \times k(s)n(s)$$

$$= -\tau(s)b(s) - k(s)t(s)$$

Theorem 3.2.5

The Frenet-Serret frames under arc length parameterization satisfies the differential equations:

$$\frac{d}{ds}\begin{bmatrix} t & n & b \end{bmatrix} = \begin{bmatrix} t & n & b \end{bmatrix}\begin{bmatrix} 0 & -k & 0 \\ k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}$$

Remark

The matrix in the previous theorem is skew-symmetric. For those who know matrix groups, it's because the Lie algebra of $SO(3)$ *is* $so(3)$*. We can also understand it via linear algebra: let* $A(s) = [t(s), n(s), b(s)]$*. Then* $AA^T = I$*. Suppose* $A' = AD$*, then take derivative to* $AA^T = I$ *we get*

$$0 = A'A^T + AA'^T = ADA^T + AD^TA^T = A(D + D^T)A^T$$

So $D + D^T = 0$*.*

Example

Example 3.2.6

Consider the curve $\alpha : \mathbb{R} \to \mathbb{R}^3$, $t \mapsto (\cos(t), \sin(t), t)$.

- Find the arc length parameterization.
- Find the Frenet-Serret frames.
- Find the curvature and torsion.

Example

Example 3.2.7

Suppose $\alpha : (-1, \infty) \to \mathbb{R}^3$ is a smooth regular curve, $\alpha' \cdot \alpha'' = 1$, $\alpha'(0) = (1, 1, 0)$.

- Find the arc length parameterization using $t_0 = 0$.
- Write down the curvature and torsion using α' and α''.

Fundamental Theorem of the Local Theory of Space Curves

Just like for plane curves, the curvature and torsion under arc length parameterization **completely charaterized the curve**.

Remark

There are some key differences between space curves and plane curves however:

- *There is no signed curvature for space curves.*
- *There is no Frenet frame, hence no torsion, when curvature is* 0*.*
- *The "normal vector" for space curves is different from the "normal vector" for plane curves.*

The statement of Fundamental Theorem need to take these into account.

Theorem 3.3.1 (Fundamental Theorem of LTSC)

Let I be an interval containing zero, k a **positive** *smooth function on I, τ a smooth function on I, then there is a smooth regular curve $\alpha : I \to \mathbb{R}^3$, unique up to rigid motion (rotation and translation), which is an arc length parameterization, and the curvature at $\alpha(t)$ is $k(t)$ and the torsion at $\alpha(t)$ is $\tau(t)$.*

Key idea of the proof: Picard's theorem for ODE.

Picard's Theorem

Theorem 3.3.2 (Picard's Theorem)

Let I be any interval, $t_0 \in I$, $x_0 \in \mathbb{R}^n$, F is a function from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ that is continuous and Lipschitz on the second parameter (i.e. there is some number L such that $\|F(t,x) - F(t,y)\| \leq L\|x - y\|$), then the initial value problem

$$y(t_0) = x_0, y' = F(t, y)$$

has a unique solution. Furthermore, if F is smooth, or real analytic, so is the solution.

Example 3.3.3

$$y_0' = -y_1, y_1' = y_0, y_0(0) = y_1(0) = 1$$

Has a unique solution $y_0(t) = \sqrt{2}\cos(t + \pi/4), y_1(t) = \sqrt{2}\sin(t + \pi/4)$.

Review for last lecture

- Frenet-Serret frames.
- Frenet equations.
- Fundamental Theorem of LTSC
- Picard's Theorem

Proof of Fundamental Theorem of LTSC.

We prove it in the case when k and τ are bounded. The argument for unbounded k and τ is a standard trick in analysis.
Firstly, we prove existence: consider the system of differential equations:

$$t(0) = e_1, n(0) = e_2, b(0) = e_3, t' = kn, n' = -kt - \tau b, b' = \tau n$$

Here we see the above as an IVP for a function $I \to \mathbb{R}^9$,
$s \mapsto (t(s), n(s), b(s))$.
Because both k and τ are smooth hence bounded, the equation is Lipschitz, hence satisfies the assumption of Picard's Theorem, hence there is a unique solution on I.

Proof of Fundamental Theorem of LTSC.

Now we show that for any $s \in I$, $t(s)$, $n(s)$ and $b(s)$ form an orthonormal basis of positive orientation.

- To show that they form orthonormal basis, we just need to show $[t, n, b][t, n, b]^T = I$. When $s = 0$ this is true. Take derivative on the left hand side with respect to s, we get:

$$([t, n, b][t, n, b]^T)' = [t', n', b'][t, n, b]^T + [t, n, b][t', n', b']^T$$

$$= [t, n, b]\begin{bmatrix} 0 & -k & 0 \\ k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}[t, n, b]^T + [t, n, b]\begin{bmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}[t, n, b]^T$$

$$= 0$$

- From the above, $\det([t, n, b])$ is ± 1. However this is continuous, and is 1 when $s = 0$, hence it must always be 1.

As a consequence, $b(s) = t(s) \times n(s)$.

Proof of Fundamental Theorem of LTSC.

Now let $\alpha(s) = \int_0^s t(r)dr$. We need to verify that

- α is an arc length parameterization.
- The curvature at $\alpha(s)$ is $k(s)$, torsion at $\alpha(s)$ is $\tau(s)$.

To show that α is arc length parameterization, calculate the arc length function using $t_0 = 0$, we get:

$$s(t) = \int_0^t \|\alpha'(r)\|dr = \int_0^t \|t(r)\|dr = t$$

Hence the arc length parameterization is $\alpha \circ s^{-1} = \alpha \circ id = \alpha$.

Proof of Fundamental Theorem of LTSC.

Now we calculate curvature and torsion: the curvature at $\alpha(s)$ is

$$\|\alpha''(s)\| = \|t'(s)\| = \|k(s)n(s)\| = k(s)$$

Hence $n(s)$ is indeed the unit normal vector, the torsion is

$$\|(t \times n)'\| = \|b'\| = \tau(s)$$

The existence is proved.

Proof of Fundamental Theorem of LTSC.

Now we prove the uniqueness: if α_1 is another smooth regular curve such that it is an arc length parameterization, and the curvature at $\alpha_1(s)$ is $k(s)$, torsion is $\tau(s)$. Do rigid motion to move $\alpha_1(0)$ to 0 (via a translation), and the unit tangent, unit normal and unit binormal vectors at $\alpha_1(0)$ to the three standard basis vectors (via a rotation). Then the unit tangent, normal and binormal vectors satisfies Frenet's equations, hence by Picard's theorem, they must be identical to those of α. Hence α_1 is identical to α. □

Example 3.3.4

Find arc length parameterization $\alpha : \mathbb{R} \to \mathbb{R}^3$, such that $k(s) = 1, \tau(s) = -1$.
We get system of equations

$$t(0) = (1,0,0), n(0) = (0,1,0), b(0) = (0,0,1), t' = n, n' = -t+b, b' = -n$$

Solving it, we get

$$t(s) = (\frac{\cos(\sqrt{2}s)+1}{2}, \frac{\sin(\sqrt{2}s)}{\sqrt{2}}, \frac{1-\cos(\sqrt{2}s)}{2})$$

$$n(s) = (-\frac{\sin(\sqrt{2}s)}{\sqrt{2}}, \cos(\sqrt{2}s), \frac{\sin(\sqrt{2}s)}{2})$$

$$b(s) = (\frac{1-\cos(\sqrt{2}s)}{2}, -\frac{\sin(\sqrt{2}s)}{\sqrt{2}}, \frac{1+\cos(\sqrt{2}s)}{2})$$

Summary of FTLTSC

- A smooth regular space curve with non zero curvature is, up to rigid motion, completely determined by the curvature and torsion under arc length parameterization.
- Any pair of smooth functions k, τ, $k > 0$, can be the curvature and torsion of a smooth regular space curve under arc length parameterization.
- The uniqueness is due to Frenet equations with fixed initial value has a unique solution. The existence is due to the existence of such a solution. Both follows from Picard's theorem.

Application: Local canonical forms

Similar to the LCF of plane curves, if we choose the coordinate chart such that $\alpha(0) = 0$, $t(0) = e_1$, $n(0) = e_2$, $b(0) = e_3$, and then do Taylor expansion of $\alpha(s)$ up to s^3 term, we get

$$\alpha(s) = s(1,0,0) + s^2(a_1, a_2, a_3) + s^3(b_1, b_2, b_3) + O(s^4)$$

Hence

$$t(s) = \alpha'(s) = (1,0,0) + 2s(a_1, a_2, a_3) + 3s^2(b_1, b_2, b_3) + O(s^3)$$

$$\alpha''(s) = 2(a_1, a_2, a_3) + 6s(b_1, b_2, b_3)$$

So $a_1 = a_3 = 0$, $a_2 = k/2$.

$$\alpha''' = (kn)' = k'n - k^2 t - k\tau b$$

$$\alpha'''(s) = 6(b_1, b_2, b_3)$$

Hence $b_1 = \frac{-k^2}{6}$, $b_2 = \frac{k'}{6}$, $b_3 = \frac{-k\tau}{6}$. This is called the **local canonical form.**

Applications of LCF

LCF

$$\left(s - \frac{k^2}{6}s^3 + O(s^4), \frac{k}{2}s^2 + \frac{k'}{6}s^3 + O(s^4), -\frac{k\tau}{6}s^3 + O(s^4)\right)$$

Remark

- *When $k > 0$, there is a neighborhood of p on the curve where all points on the curve within this neighborhood lies within a half space whose boundary pass through p.*
- *The plane spanned by t and n, passing through $p = \alpha(0)$, has the property that distance from $\alpha(s)$ to the plane is $O(s^3)$, and this is the unique plane that satisfies this condition. Hence it should be said to be the* **osculating plane**.

Example

Example 3.3.5

α is a smooth regular space curve under arc length parameterization, which lies on the unit sphere. What can we say about $k(s)$ and $\tau(s)$, assuming $k' \neq 0$, $\tau \neq 0$?

- α lies on the unit sphere, hence $\alpha \cdot \alpha = 1$. Hence

$$\alpha' \cdot \alpha = 0$$

$$\alpha'' \cdot \alpha = -\alpha' \cdot \alpha' = -1$$

$$\alpha''' \cdot \alpha = -\alpha'' \cdot \alpha' = 0$$

- $t = \alpha'$, $kn = \alpha''$, $k'n - k^2 t - k\tau b = \alpha'''$, so $k(n \cdot \alpha) = -1$, $k'(n \cdot \alpha) - k\tau(b \cdot \alpha) = 0$.

- Hence, the coordinate of α under the chart (t, n, b) must be $(0, -1/k, -\frac{k'}{k^2\tau})$. However, $\|\alpha\|^2 = 1$, hence

$$\frac{1}{k^2} + \frac{k'^2}{k^4\tau^2} = 1$$

Can you prove that this is a sufficient condition for α to be on a sphere of radius 1? This is exercise 13 of Section 1.5.

Review for LTSC

- Smooth regular space curve
- Arc length function, arc length parameterization
- Unit tangent, normal, binormal vectors, curvature, torsion
- **Frenet equations**
- Fundamental theorem
  - When $k > 0$, a curve is, up to rigid motion, uniquely determined by k and τ under arc length parameterization.
  - Any smooth function $k > 0$, τ can be the curvature and torsion of a smooth regular space curve under arc length parameterization.

4. Hypersurfaces
   - Review for implicit function theorem
   - Definition, coordinate charts
   - Tangent space
   - First fundamental form, geometry on surfaces
   - Orientability
   - Gauss Map, Second fundamental form

Review for implicit function theorem

Theorem 4.1.1 (Implicit function theorem)

Let U be an open subset of $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$, $f : U \to \mathbb{R}^m$ a C^1 function. Suppose $f((a,b)) = y$, and the last m columns of the Jacobian of f at (a,b) is linearly independent, then there is some open neighborhood V of a, C^1 function g from V to $\mathbb{R}^m$, such that $b = g(a)$, and for any $x \in V$, $f(x, g(x)) = y$.

Remark

When $m = n$, $f(a,b) = b - h(a)$, this becomes the **inverse function theorem**.

Definition

Definition 4.2.1

A **smooth regular function** on an open set of $\mathbb{R}^n$ is a smooth function where the derivative is nowhere zero.

Definition 4.2.2

A subset S of $\mathbb{R}^n$ is a **smooth regular hypersurface** if for every $p \in S$, there is an open neighborhood $p \in U$, a smooth regular function f on U such that S is a level set of f.

This definition, due to the argument in 2.2 in the textbook, is equivalent to the definition in Section 2.2 of the textbook.

Example 4.2.3

The followings are smooth regular hypersurfaces in $\mathbb{R}^2$:

- $\{(x, y) : x^2 - y^2 = 1\}$
- $\{(t, \sin(t)) : 0 < t < 1\}$

The followings are not smooth regular hypersurfaces in $\mathbb{R}^2$:

- $\{(t\cos(1/t), t\sin(1/t)) : 0 < t < 1\} \cup \{(0, 0)\}$
- $\{(x, y) : x^2 - y^2 = 0\}$

Some general definitions regarding topology of $\mathbb{R}^n$

Definition 4.2.4

- A subset U of $\mathbb{R}^n$ is called **open**, if for any $p \in U$, there is some ball centered at p with radius $r > 0$ contained in U; it is called **closed** if its complement is open.
- Let $A \subset \mathbb{R}^n$, a subset of A is called **open or closed in** A, if it is the intersection between A and some open set or closed set in $\mathbb{R}^n$.
- A map is called continuous if the preimage of open set is open.
- A continuous bijection whose inverse is also continuous is called a **homeomorphism**, a smooth bijection whose inverse is smooth is called a **diffeomorphism**.

Coordinate charts

Some immediate consequences of the implicit function theorem applied to smooth regular hypersurface:

- Let S be a smooth regular hypersurface. $p = (p_1, \ldots, p_n) \in S$ be a point. Suppose that around p, S is defined as the level set of some smooth regular function f.
- Because f is regular, we can find a smaller open ball B centered at p where one of the partial derivatives of f is always non-zero. Suppose it is $\partial_n f \neq 0$.
- Now apply implicit function theorem, we get $g : V \to \mathbb{R}$, where V is a neighborhood of $p' = (p_1, \ldots, p_{n-1})$, $g(p') = p_n$, and for all $x \in V$, $(x, g(x)) \in S$. Define $i : V \to S$ to be $i(x) = (x, g(x))$.
- Now look at open set $U = B \cap (V \times \mathbb{R})$, then by mean value theorem, we know that $S \cap U = i(V)$, and i is a homoemorphism from V to $S \cap U$, because i^{-1} can be written as an orthogonal projection which is continuous.
- i is evidently smooth with rank of Jacobian $n-1$.

The map i is a special case of the following:

Definition 4.2.5

A **coordinate chart**, or **parameterization**, of a hypersurface $S \subset \mathbb{R}^n$ is a **smooth homeomorphism** from some open set $U \subset \mathbb{R}^{n-1}$ to an open subset set of S, whose Jacobian matrix is **always of rank** $n-1$. The image is called a **coordinate neighborhood**.

And the argument above implies that

Theorem 4.2.6

Around every point in a smooth regular hypersurface there is a coordinate neighborhood.

Example 4.2.7

Let $S = \{x^2 + y^2 + z^2 = 1\}$, find a coordinate chart around $(1, 0, 0)$.

Coordinate change

Theorem 4.2.8 (Proposition 1 in Section 2.3)

Let S be a smooth regular hypersurface, $i_1 : V_1 \to S$, $i_2 : V_2 \to S$ be two coordinate charts, $i_1(V_1) \cap i_2(V_2) \neq \emptyset$, then $i_2^{-1} \circ i_1 : i_1^{-1}(i_1(V_1) \cap i_2(V_2)) \to i_2^{-1}(i_1(V_1) \cap i_2(V_2))$ is a diffeomorphism.

Definition 4.2.9

This diffeomorphism $i_2^{-1} \circ i_1$ is called a **coordinate change**.

Proof idea

These maps are obviously homeomorphisms, hence we only need to show that they are smooth. Let p be a point in the intersection of two coordinate neighborhoods, one can build a coordinate chart $i_p : V_p \to S$ using implicit function as before, then use inverse function theorem to show that $i_2^{-1} \circ i_p$ is smooth. i_p^{-1} is orthogonal projection, hence $i_p^{-1} \circ i_1$ is also smooth, hence $i_2^{-1} \circ i_p \circ i_p^{-1} \circ i_1$ is smooth on $i_1^{-1}(i_1(V_1) \cap i_2(V_2) \cap i_p(V_p))$.

More definitions

Definition 4.2.10

A differentiable manifold is a set M, a collection of subsets $U \subset M$, each corresponding to a bijection i from an open subset of $\mathbb{R}^n$, such that for any two such bijections i and i', $i'^{-1} \circ i$ is smooth where it is defined. Usually we add a few more technical assumptions from point set topology which we will ignore for now.

It is easy to see that the charts we defined using implicit function theorem gives a differentiable manifold structure to a smooth regular hypersurface.

Review for Previous lecture

- Definition of smooth regular hypersurfaces.
- Concept of coordinate charts
- Existence of charts via implicit function theorem (proof sketch: definition->IFT->MVT)
- Change of coordinates. (follow the textbook, or go through charts we found earlier)

Example 4.2.11

Find some coordinate charts on the unit circle $\{(x, y) : x^2 + y^2 = 1\}$ and find the change of coordinate functions.

Definition 4.2.12

- A **smooth function** on a smooth regular hypersurface S is a function f, such that for any $p \in S$, there is a coordinate chart $i : V \to S$, such that $f \circ i$ is smooth.
- A **smooth map** between two smooth regular hypersurfaces S and S' is a map f, such that for any $p \in S$, there are coordinate charts $i : U \to S$, $j : V \to S'$, smooth map $g : U \to V$, such that $f \circ i = j \circ g$.

Theorem 4.2.13

A map from a smooth regular hypersurface in $\mathbb{R}^m$ to a smooth regular hypersurface in $\mathbb{R}^n$ is smooth iff it is of the form $x \mapsto (f_1(x), \ldots, f_n(x))$, where f_i are smooth functions.

One can prove this using the charts obtained via implicit function theorems.

Tangent Spaces

Definition 4.3.1

Let S be a smooth regular curve, the tangent space of S at p, denoted as $T_p(S)$, can be defined in one of the three ways:

1. Suppose locally S is a level set of some smooth regular function f, then $T_p(S)$ is the null set (or kernel) of the Jacobian matrix of f at p.
2. Suppose there is a local coordinate chart i around p, then $T_p(S)$ is the column space (or range) of the Jacobian matrix of i at $i^{-1}(p)$.
3. $T_p(S)$ is the set of all possible $\alpha'(0)$, where α is a smooth curve on S and $\alpha(0) = p$.

Theorem 4.3.2

The three definitions of $T_p(S)$ are all equivalent.

Proof.

We denote the three definitions as T_1, T_2 and T_3.

- $T_1 = T_2$: both T_1 and T_2 are $n-1$-dimensional subspace of $\mathbb{R}^n$. $f \circ i$ is constant, hence $J(f)J(i) = 0$, hence $T_2 \subset T_1$, hence $T_1 = T_2$.
- $T_2 \subset T_3$: for any $v = (a_1, \ldots, a_n)$, consider $\alpha(t) = i(i^{-1}(p) + vt)$, then $\alpha'(0)$ is the linear combination of the columns of $J(i)$ with coefficients a_i.
- $T_3 \subset T_1$: $f \circ \alpha$ is constant, hence $\alpha' \subset kerJ(f)$.

□

Review for last lecture

Key properties and concepts of smooth regular surfaces:

- Smooth regular hypersurfaces, coordinate charts, smooth functions and smooth maps
- Existence of coordinate charts: IFT plus MVT
- Existence of smooth change of coordinate
- Three equivalent definitions of the tangent space.

Depending on the context, sometimes "tangent space" or "tangent plane" at p is the affine space $p + T_p(M)$.

The **normal space** $N_p(S)$ is defined as the orthogonal complement of the tangent space.

Example 4.3.3

$t \mapsto (\cos(t)/2, \sin(t)/2, \sqrt{3}/2)$ is a smooth regular curve on $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$.

Maps between tangent spaces

- Coordinate charts gives bijections between $T_p(S)$ and $\mathbb{R}^{n-1}$, which we can also call $T_{i^{-1}(p)}(V)$, which is represented by the Jacobian map.
- Smooth map between hypersurfaces induce linear maps between tangent spaces, via the second or third definition.

Example 4.3.4

Let $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Show that $(x, y, z) \mapsto (-x, -y, -z)$ is smooth, and find the induced map between two tangent spaces.

Quadratic forms

Recall that a **symmetric bilinear form** on a vector space V is a map $V \times V \to \mathbb{R}$, denoted as $(x, y) \mapsto \langle x, y \rangle$, which satisfies:

$$\langle x, y \rangle = \langle y, x \rangle$$

$$\langle ax_1 + bx_2, y \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle$$

If $\{x_i\}$ is a finite basis of V, then

$$\langle \sum_i a_i x_i, \sum_i b_i x_i \rangle = [a_1 \dots a_n] A [b_1 \dots b_n]^T$$

Where A is a symmetric matrix.

A **quadratic form** on V is $Q : V \to I$ defined by $Q(x) = \langle x, x \rangle$ for some symmetric bilinear form $\langle , \rangle$. Quadratic forms and symmetric bilinear forms have a 1-1 correspondence via

$$Q(x) = \langle x, x \rangle$$

$$\langle x, y \rangle = \frac{1}{4}(Q(x + y) - Q(x - y))$$

First fundamental form

Definition 4.4.1

The first fundamental form is a quadratic form on $T_p(S)$ defined via Euclidean inner product in $\mathbb{R}^n$.

The first fundamental form allow us to do geometry on hypersurface S, in particular, define concepts like **distance, angle and area**. Later we will show that first fundamental form characterizes the **intrinsic geometry** of the hypersurface. This is also called the Riemannian metric.

Review of last lecture

- Maps between tangent spaces
- First fundamental form

First fundamental form in coordinate chart

Because we want to use $\times$ to simplify our notations, from now on we will restrict ourselves to the $n = 3$ case, i.e. surfaces in 3-d Euclidean spaces, but everything can be easily generalized to arbitrary n.
Now, under coordinate chart $i : V \to S$, $T_p(S)$ is spanned by x_u and x_v which are the image of two basis vectors under Ji, and the first fundamental form becomes

$$I(ax_u + bx_v) = Ea^2 + 2Fab + Gb^2$$

And the corresponding inner product is

$$\langle ax_u + bx_v, a'x_u + b'x_v \rangle = Eaa' + F(ab' + a'b) + G(bb')$$

Arc length

Let α be a smooth regular curve on a smooth regular surface S. Assume that α lies within a coordinate neighborhood (if not, subdivide it). Then α can be written as $i \circ \beta$, where $\beta = (\beta_1, \beta_2)$ is a smooth regular curve on V. Then the length of α is

$$\int_I \|\alpha'\| dr = \int_I I(\alpha')^{1/2} dr$$

$$= \int_I I(\beta_1' x_u + \beta_2' x_v)^{1/2} dr$$

$$= \int_I (E\beta_1'^2 + 2F\beta_1'\beta_2' + G\beta_2'^2)^{1/2} dr$$

To measure the distance between two points on S, find the shortest smooth regular curve between them and measure its length.

Angles and area

- The angle θ between two tangent vectors x and y is $\cos^{-1}\left(\frac{\langle x,y\rangle}{(I(x)I(y))^{1/2}}\right)$.
- The area of a shape that lies in a coordinate neighborhood, of the form $i(D)$ where $D \subset V$, is defined as

$$\int_D \|x_u \times x_v\| dudv = \int_D \sqrt{\begin{vmatrix} \langle x_u, x_u\rangle & \langle x_u, x_v\rangle \\ \langle x_v, x_u\rangle & \langle x_v, x_v\rangle \end{vmatrix}} dudv$$

- For D that can not fit inside a single coordinate neighborhood, split it and add the calculated areas.

Example 4.4.2

Let $S = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$

- Write down the first fundamental form on S using a coordinate chart.
- Find the angle between two circles on S.
- Find the total area of S.

Application of coordinate charts: Map projections

- Orthographic projection (our IFT charts) (Hipparchus of Nicaea, 2nd c. BCE)
- Stereographic projection (Hipparchus of Nicaea, 2nd c. BCE)
- Equirectangular Projection (Marinus of Tyre, 1-2 c. CE)
- Mercator Projection (cylinder, conformal) (1569)
- Lambert Projection (from axis) (1772)

Can you write down the first fundamental form under all these local coordinate charts?

Review for last lecture

- First fundamental form
- Length, angle, area
- First fundamental form under coordinate charts

Example 4.4.3

Sphere under Central Cylindrical projection Consider the coordinate chart of unit sphere: $i : (0, 2\pi) \times \mathbb{R} \to S^2$,
$i(u, v) = (\cos u/\sqrt{1+v^2}, \sin u/\sqrt{1+v^2}, v/\sqrt{1+v^2})$. Write down the first fundamental form under this coordinate chart.

Orientability of surfaces

Definition 4.5.1

A smooth regular hypersurface S is called orientable, if it admits a set of coordinate charts covering S, where the Jacobian of all change of coordinates are orientation preserving.

Theorem 4.5.2

Let S be a smooth regular surface, then S is orientable, if and only if there is a continuous, non zero map N from S to $\mathbb{R}^3$, such that $N(p) \in N_p(S)$.

Proof.

If S is orientable, we can define $N(p) = \frac{x_u \times x_v}{\|x_u \times x_v\|}$.
If N is such a function, we choose a set of coordinate charts covering S, and precompose $(u, v) \mapsto (v, u)$ for any chart where $x_u \times x_v$ and N are not in the same direction. □

Example 4.5.3

- If S can be covered by a single coordinate chart, then S is orientable.
- The unit sphere is orientable,
- Any smooth regular hypersurface in $\mathbb{R}^2$ is orientable.
- The Mobius strip is not orientable.

When discussing local geometry of a surface, we can always assume that it is orientable.

Vector bundles and sections

Definition 4.5.4

- If S is a smooth regular hypersurface, on every point of S, there is a subspace V of $\mathbb{R}^n$ of dimension d, and the subspace changes smoothly on S (are defined by linear equations with smooth coefficients). Then we call the set $E = \{(p, v) \in S \times \mathbb{R}^n : v \in V(p)\}$ a **vector bundle** on S. The subspace $V(p)$ is called the **fiber** at p, denoted as E_p.
- A smooth map $f : S \to \mathbb{R}^n$, such that $f(p) \in E_p$, is called a **section** of a vector bundle E.

Example 4.5.5

- The vector bundle $\{(p, v) : v \in T_p(S)\}$ is called the **tangent bundle**, denoted as $T(S)$.
- The vector bundle $\{(p, v) : v \in N_p(S)\}$ is called the **normal bundle**, denoted as $N(S)$.
- The bundle $S \times \mathbb{R}^n$ is denoted as $T(R^n)|_M$.

Now the theorem above becomes

Theorem 4.5.6

$S \subset \mathbb{R}^3$ is a smooth regular surface, then S is orientable iff $N(S)$ has a non-zero section.

Midterm Review

- Definition of smooth regular plane curve and smooth regular space curves
- Arc length function and arc length parameterization
- Geometry of plane curves
  - Unit tangent and normal vector
  - Signed curvature k, $t' = kn, n' = -kt$
  - Fundamental theorem, local canonical form: $(s - k^2s^3/6 + O(s^4), ks^2/2 + k's^3/6 + O(s^4))$
- Geometry of space curves
  - Unit tangent, normal and binormal vector
  - Curvature k and torsion τ, $t' = kn, n' = -kt - \tau b, b' = \tau n$
  - Fundamental theorem, local canonical form: $(s - k^2s^3/6 + O(s^4), ks^2/2 + k's^3/6 + O(s^4), -k\tau s^3/6 + O(s^4))$.

- Definition of smooth regular hypersurfaces. When $n = 3$ we call them "surfaces".
- Coordinate charts
  - Definition: smooth homeomorphism to open subsets, Jacobian full rank.
  - Existence: Implicit function theorem
  - Change of coordinate smooth: inverse function theorem
  - Smooth functions and smooth maps
- Tangent spaces
  - Three equivalent definitions
  - Coordinate chart induces bijections from tangent spaces to R^{n-1}
  - Smooth maps induces maps between tangent spaces
  - Normal spaces
- First fundamental form
  - Definition
  - Expression under coordinate charts
  - Length, angle, area

Review examples

- Let f be a smooth real valued function. Consider smooth regular surface $S = \{(x, y, f(x, y))\}$.
    - Find a coordinate chart $i : \mathbb{R}^2 \to S$, such that $i^{-1}(x, y, z) = (x, y)$.
    - Find $T_{i(0)}(S)$ and write down the first fundamental form under the coordinate chart above.
    - Suppose α is a smooth regular plane curve under arc length parameterization. Write down the arc length function of $i \circ \alpha$.
    - Write down the curvature at $i \circ \alpha(0)$. When is this curvature equals the curvature of $\alpha(0)$?
- Show that if S is a smooth regular surface, for any $p \in S$, the normal space $N_p(S)$, as a linear subspace of $\mathbb{R}^3$, contains p, and any two points in S are connected by a smooth path, then S is contained in a sphere.
- Suppose S is a smooth regular surface that admits a coordinate chart $(u, v) \mapsto f(u) + g(v)$, which covers the whole surface. Find a curve on S along which the tangent space all contains a specific non-zero vector.

Digression: How to read and write math

- How to read definitions and propositions: figure out terms.
  Example: A coordinate chart is a smooth homeomorphism from an open set in $\mathbb{R}^{n-1}$ to an open subset of a hypersurface, such that the Jacobian has rank $n-1$.
  Example: We call a function f on a hypersurface S smooth, if for any point $p \in S$, there is a coordinate chart $i : U \to S$, $p \in i(U)$, and $f \circ i$ is smooth.
- How to read proofs: fill in gaps.
  Example: Existence of local coordinate charts.
- Understanding via examples.
  Example: Implicit Function Theorem
- How to write proofs: plan, then fill in gaps.
  Example: Show that if f is a smooth function on S, $i : U \to S$ is any coordinate chart, then $f \circ i$ is smooth.

Review for last lecture

- Tangent bundle and Normal bundle of surfaces, sections.
- Orientability
- Orientability equivalent to non-zero sections on normal bundles.

Gauss Map, Second Fundamental Form

If S is a smooth orientable regular surface, by an "orientation" we mean a set of coordinate charts covering S where change of coordinates are all orientation preserving.

Definition 4.6.1

If S is a smooth orientable regular surface in $\mathbb{R}^3$. Let S^2 be the unit sphere, then the map $G : S \to S^2$ defined locally by $p \mapsto \frac{x_u \times x_v}{\|x_u \times x_v\|}$ is called the **Gauss map**. The induced map on tangent space DG is called the **shape operator** or **Weingarten map**.

Theorem 4.6.2

Let DG be the map between tangent spaces induced by the Gauss map. Then DG is a map from $T_p(S)$ to $T_p(S)$, and is represented by a symmetric matrix under an orthonormal basis under the first fundamental form, i.e. $x \cdot DG(y) = DG(x) \cdot y$.

Proof.

DG is a map from $T_p(S)$ to $T_{G(p)}(S^2)$, and both are orthogonal complements of $G(p)$ hence equal. Now we consider a local coordinate chart i whose image contains p, (bi)-linearility implies that we only need to check the identity for x and y being $x_u = i_u$ and $x_v = i_v$, and the only case one need to check is $x = x_u$ and $y = x_v$.
Now because $G \cdot x_u = G \cdot x_v = 0$, we have
$G \cdot i_{uv} + G_v \cdot x_u = G \cdot i_{vu} + G_u \cdot x_v = 0$. By definition it's easy to see $G_u = DG(x_u)$, $G_v = DG(x_v)$. The identity now follows from $i_{uv} = i_{vu}$ which we know from multivariable calculus. □

Example 4.6.3

Find the map DG of a point in a:

- Planes
- Cylinders
- Cones
- Spheres

Definition 4.6.4

The **second fundamental form** on $T_p(S)$ is defined as

$$II(v) = -\langle v, DG(v) \rangle$$

Recall from linear algebra:

Theorem 4.6.5

Any symmetric $n \times n$ matrix has n eigenvectors forming an orthonormal basis, and all eigenvalues are real.

Review from last lecture

- Gauss map: $G(p) = \frac{x_u \times x_v}{\|x_u \times x_v\|}$
- Shape operator: $DG : T_p(S) \to T_p(S)$.
- Shape operator is self adjoint.
- Second fundamental form $II(v) = -v \cdot DG(v)$.

Midterm Exam:

Definition 4.6.6

- The negative of the eigenvalues of DG are called **principal curvatures**. The eigenvectors are called **principal directions**.
- The negative of the average of the two eigenvalues is called the **mean curvature**. A surface whose mean curvature vanishes is called a **minimal surface**.
- The product of the two eigenvalues is called the **Gaussian curvature**.
- The non zero vectors v where II vanishes is called an **asymptotic direction**.

In many other textbooks there isn't a negative sign, and mean curvature and principal curvatures are also off by a sign. We follow the convention of Ddo Carmo.

Shape operator and second fundamental form under local coordinate

- Starting data: $i : U \to S$.
- First fundamental form: $x_u = i_u$, $x_v = i_v$, matrix representation of first fundamental form $A = [x_u, x_v]^T [x_u, x_v]$.
- Gauss map $G \circ i = \frac{x_u \times x_v}{\|x_u \times x_v\|}$. $[DG(x_u), DG(x_v)] = J(G \circ i)$. Second fundamental form: $-[x_u, x_v]^T [DG(x_u), DG(x_v)] = -[x_u, x_v]^T J(G \circ i)$.
- To find matrix representation B of the shape operator, we have

$$[x_u, x_v]B = [DG(x_u), DG(x_v)]$$

$$B = (J(i)^T J(i))^{-1} J(i)^T J(G \circ i)$$

Remark

The matrix representation of the second fundamental form of surface is often denoted as $\begin{bmatrix} e & f \\ f & g \end{bmatrix}$.

Example 4.6.7

Calculate the shape operator and second fundamental of the surface $\{(x, y, x^2 + y^2)\}$ at $(0, 0, 0)$.

Geometric meaning and generalization

- Normal curvature: If α is a smooth regular curve on S under arc length parameterization, then the **normal curvature** is $\alpha'' \cdot n = -\alpha' \cdot DG\alpha'$.
  - Principal directions are where normal curvature is at maximum or minimum.
  - Asymptotic directions are where normal curvature is zero.
- Gaussian curvature: ratio of signed area.
- Generalization of Gauss map, shape operator, second fundamental form in other dimensions. For hypersurfaces in $\mathbb{R}^2$, the "Gaussian curvature" is the signed curvature.

The norm of the projection of α'' on $T_p(S)$ is the absolute value of the **geodesic curvature**, which we will discuss in the next chapter.

Review

- Hypersurfaces, coordinate charts, coordinate changes
- First fundamental form, length, angle and area
- Shape operator and second fundamental form, principal curvature, mean curvature, Gaussian curvature.
- Geometric meaning of the second fundamental form: normal curvature, area ratio.

5 Intrinsic geometry of surfaces
- Parallel transport
- Intrinsic curvature, Gauss theorem
- Geodesics

Definition 5.0.1

A map between hypersurfaces is an isometry if it preserves distance, a conformal map if it preserves angles.

- "Local isometry" means an isometry from a neighborhood of one point on one surface to a neighborhood of another point on another surface.
- Properties invariant under isometry are called "intrinsic".
- If a property can be evaluated using only the local formula or first fundamental form, then it is intrinsic.

Parallel Transport

From first fundamental form we can define angles, and after integration, areas. However to do more "global" geometry we need to connect tangent spaces at one point to tangent spaces at another. This geometric object is called **connection**, and in particular, we will talk about **Levi-Civita Connection** or **Parallel Transport**.

Definition

Definition 5.1.1

Let S be a smooth regular surface, $i : U \to S \subset \mathbb{R}^3$ a coordinate chart, $\alpha : I \to i(U)$, $0 \in I$ is a smooth regular path. Then the parallel transport is a family of linear transformations $P_{\alpha,t} : T_{\alpha(0)}(S) \to T_{\alpha(t)}(S)$, such that $\frac{d}{dt} P_{\alpha,t}(x)$ is in $N_{\alpha(t)}(S)$.

As a consequence, P preserves length, i.e. the first fundamental form. It is easy to see that reparameterization of α sends parallel transport to parallel transport.

Review for last lecture

- Geometric meaning of second fundamental form: normal curvature, generalization to other dimensions, area.
- Isometric and conformal maps
- Definition of parallel transport along a smooth regular curve.

Calculation

Theorem 5.1.2

The Parallel transport under basis x_u, x_v is uniquely determined by $i^{-1} \circ \alpha$ and the first fundamental form under coordinate chart i. In other words, it **exists**, *is* **unique**, *and also* **intrinsic**.

Proof.

Suppose $[P_{\alpha,t}(x_u), P_{\alpha,t}(x_v)] = [x_u, x_v]A(t)$. It is evident that $A(0) = I_2$. Let $\beta = i^{-1} \circ \alpha = (\beta_1, \beta_2)$. Now take derivative with respect to t, we get

$$[\partial_u x_u \beta_1' + \partial_v x_u \beta_2', \partial_u x_v \beta_1' + \partial_v x_v \beta_2']A(t) + [x_u, x_v]A'(t)$$

Proof, cont.

By assumption, both columns must be in $N_{\alpha(t)}(S)$, hence multiplying with $[x_u, x_v]^T$ from the left must be zero. Hence:

$$[x_u, x_v]^T[x_u, x_v]A'(t) = -[x_u, x_v]^T[\partial_u x_u \beta_1' + \partial_v x_u \beta_2', \partial_u x_v \beta_1' + \partial_v x_v \beta_2']A(t)$$

$$= (-\beta_1'([x_u, x_v]^T \partial_u [x_u, x_v]) - \beta_2'([x_u, x_v]^T \partial_v [x_u, x_v]))A(t)$$

The existence and uniqueness of parallel transport now follows from Picard's theorem.

We replace u by 1 and v by 2, and $p, q, r \in \{1, 2\}$. $x_p^T \partial_q x_r$ is also denoted as Γ_{pqr}, called **Christoffel symbols of the first kind**.

Proof, cont.

Now we prove that the Christoffel symbols can be represented using only the first fundamental form, i.e. the matrix $[x_u, x_v]^T[x_u, x_v]$. For any $p, q, r \in \{u, v\}$,

$$\frac{1}{2}(\partial_q(x_p^T x_r) + \partial_r(x_p^T x_q) - \partial_p(x_q^T x_r))$$

$$= \frac{1}{2}((i_p \cdot i_r)_q + (i_p \cdot i_q)_r - (i_q \cdot i_r)_p)$$

$$= \frac{1}{2}(i_{pq} \cdot i_r + i_p \cdot i_{qr} + i_{pr} \cdot i_q + i_p \cdot i_{qr} - i_{pq} \cdot i_r - i_q \cdot i_{pr})$$

$$= i_p \cdot i_{qr} = x_p^T \partial_q x_r = x_p^T \partial_r x_q$$

Now the intrinsicness is proved. □

Example 5.1.3

Find the Christoffel symbols of the first kind, and parallel translation along a circle, on unit spheres.

Consider coordinate chart

$$i : \{(u, v) : -\pi < u < \pi, -\pi/2 < v < \pi/2\} \to S^2$$

$$i(u, v) = (cosv \cos u, \cos v \sin u, sinv)$$

Then

$$x_u = (-\sin u \cos v, \cos u \cos v, 0), x_v = (-\sin v \cos u, -\sin v \sin u, \cos v)$$

$$\partial_u x_u = (-\cos u \cos v, -\sin u \cos v, 0)$$

$$\partial_v x_u = \partial_u x_v = (\sin u \sin v, -\cos u \sin v, 0)$$

$$\partial_v x_v = (-\cos u \cos v, -\sin u \cos v, -\sin v)$$

$$x_u \cdot x_u = \cos^2 v, x_u \cdot x_v = 0, x_v \cdot x_v = 1$$

Hence the Christoffel sysmbols are

$$\Gamma_{111} = 0, \Gamma_{211} = \sin v \cos v$$

$$\Gamma_{122} = 0, \Gamma_{222} = 0$$

$$\Gamma_{121} = \Gamma_{112} = -\cos v \sin v, \Gamma_{212} = \Gamma_{221} = 0$$

Alternatively, we can do the computation using first fundamental forms, e.g.

$$\Gamma_{112} = \frac{1}{2}(\partial_u 0 + \partial_v \cos^2 v - \partial_u 0) = -\cos v \sin v$$

Now consider the path $\alpha : (-\pi, \pi) \to S$,
$\alpha(t) = (\cos r \cos t, \cos r \sin t, \sin r)$. Then $\beta = i \circ \alpha$ is $\beta(t) = (t, r)$. Hence the differential equation becomes:

$$\begin{bmatrix} \cos r^2 & 0 \\ 0 & 1 \end{bmatrix} A' = -\begin{bmatrix} 0 & -\cos r \sin r \\ \cos r \sin r & 0 \end{bmatrix} A$$

$$A' = -\begin{bmatrix} 0 & -\sin r / \cos r \\ \cos r \sin r & 0 \end{bmatrix} A$$

So, if one apply parallel transport to the vector x_u, we have
$A(t)(e_1) = (\cos(t \sin r), -\cos r \sin(t \sin r))$. When $t \to \pi$ and $t \to -\pi$, the resulting vectors have an angle of $2\pi \sin r$. When r is small, we can also made it into $2\pi(1 - \sin r)$ which equals to the area bounded by the circle α.

Remark

Parallel translation along any smooth simple closed curve on unit sphere has rotation angle equals to the area of the bounded region.

Example 5.1.4

Calculate the Christoffel symbols for

- Plane under Cartesian coordinate.
- Cylinder under coordinate chart $(u, v) \mapsto (\cos u, \sin u, v)$
- Plane under Polar coordinate.

Definition 5.1.5

The Christoffel symbols of the second kind is defined as $\partial_p x_q - \sum_r \Gamma^r_{pq} x_r \in N_P(S)$.

It is easy to see that the Christoffel symbols of the second type can be calculated using Christoffel symbols of the first kind and the first fundamental form.

Review for Levi-Civita connection

$\{u, v\}$ is represented by $\{1, 2\}$. $i : U \to S$ a coordinate chart.

- Christoffel symbols:

$$\Gamma_{ijk} = x_i \cdot \partial_j x_k = x_i \cdot \partial_k x_j$$

$$\partial_j x_k = \partial_k x_j = \sum_i \Gamma^i_{jk} x_i + LG(p)$$

- Relationship between Christoffel symbols of first and second kind:

$$\Gamma_{ijk} = \sum_l (x_i \cdot x_l) \Gamma^l_{jk}$$

- Christoffel symbols from first fundamental form

$$\Gamma_{ijk} = \frac{1}{2}(\partial_j (x_i \cdot x_k) + \partial_k (x_i \cdot x_j) - \partial_i (x_j \cdot x_k))$$

- Matrix representation of parallel transport:

$$\sum_k (x_i \cdot x_k) A'_{kj}(t) = -\sum_k \sum_l (\Gamma_{ilk} \beta'_l) A_{kj}$$

$$A'_{ij}(t) = -\sum_l (\sum_k \Gamma^i_{kl} \beta'_k) A_{lj}$$

Intrinsic (Riemann) curvature, Holonomy

- Suppose S is, or is isometric to, a plane, $i : U \to S$ a coordinate chart. Then parallel transport along any closed curve will send every vector back to itself.
- However, if you walk from the south pole to north pole via two different longitudes, and do parallel transport along the way, then it is no longer true.
- Hence, the turning angle after parallel transport along a closed curve, called **holonomy**, characterized how much a surface is **intrinsically curved**.

Theorema Egregium ("Remarkable theorem")

Theorem 5.2.1

The Gaussian curvature can be calculated using ONLY first fundamental form. In other words, Gaussian curvature is invariant under local isometry.

Proof idea

Because Gaussian curvature is smooth, it is continuous, hence at every point there is a neighborhood where the Gaussian curvature is almost constant. Find a simple closed curve in that neighborhood α. Consider the corresponding curve on unit sphere $G \circ \alpha$. Because $T_{\alpha(t)}(S) = T_{G(\alpha(t))}(S^2)$, the parallel transport along α and $G \circ \alpha$ are identical hence has the same turning angle, which equals the area bounded by $G \circ \alpha$. Hence, the turning angle of α divides the area of the part on S bounded by α is the Gaussian curvature. Hence Gaussian curvature is intrinsic because parallel transport is intrinsic.

We can make the idea more precise as follows:

Proof.

Let the matrix representation of the $-DG$ be $[b_{ij}]$, i.e. $\partial_j G = -\sum_i b_{ij} x_i$ (here G means $G \circ i$ where i is the coordinate chart).
Suppose $\partial_i x_j = \sum_k \Gamma_{ij}^k x_k + L_{ij} G$. Because $x_i \cdot G = 0$, we have $L_{ij} + x_i \cdot \partial_j G = 0$, hence $L_{ij} = \sum_k (x_i \cdot x_k) b_{kj}$ is the i, j-th entry of the matrix representation of second fundamental form. Now look at the equation $\partial_1 \partial_2 x_1 = \partial_2 \partial_1 x_1$. By calculation, the left hand side becomes

$$\partial_1(\Gamma_{21}^1)x_1 + \Gamma_{21}^1(\Gamma_{11}^1 x_1 + \Gamma_{11}^2 x_2 + eG) + \partial_1(\Gamma_{21}^2)x_2 + \Gamma_{21}^2(\Gamma_{12}^1 x_1 + \Gamma_{12}^2 x_2 + fG)$$

$$+\partial_1(f)G - f b_{11} x_1 - f b_{21} x_2$$

The right hand side is

$$\partial_2(\Gamma_{11}^1)x_1 + \Gamma_{11}^1(\Gamma_{21}^1 x_1 + \Gamma_{21}^2 x_2 + fG) + \partial_2(\Gamma_{11}^2)x_2 + \Gamma_{11}^2(\Gamma_{22}^1 x_1 + \Gamma_{22}^2 x_2 + gG)$$

$$+\partial_2(e)G - e b_{21} x_1 - e b_{22} x_2$$

Proof.

Look at the coefficients of x_2 on both sides, we get

$$eb_{22} - fb_{21} = \Gamma^1_{11}\Gamma^2_{21} + \partial_2\Gamma^2_{11} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{21}\Gamma^2_{11} - \partial_1\Gamma^2_{21} - \Gamma^2_{21}\Gamma^2_{12}$$

The right hand side is evidently intrinsic, the left hand side equals

$$(x_1 \cdot x_1)b_{11}b_{22} + (x_1 \cdot x_2)b_{21}b_{22} - (x_1 \cdot x_1)b_{12}b_{21} - (x_1 \cdot x_2)b_{22}b_{21}$$

$$= (x_1 \cdot x_1)(b_{11}b_{22} - b_{21}b_{12})$$

□

Remark

To see how the proof is related to the idea stated earlier, consider parallel transport of x_1 *along* $\beta_1(t) = (t, s)$*, and* $\beta_2(t) = (s, t)$*, one gets*

$$P_{t,\beta_1}(x_1) = x_1 - t\sum_j \Gamma^j_{11} x_j + o(t) = x_1 - t\partial_1 x_1 + teG + o(t)$$

$$P_{t,\beta_2}(x_1) = x_1 - t\sum_j \Gamma^j_{21} x_j + o(t) = x_1 - t\partial_2 x_1 + tfG + o(t)$$

The study of the infinitestimal holonomy is equivalent to calculating $\frac{d}{ds}$ *of the t-coefficients of the former minus* $\frac{d}{ds}$ *of the t-coefficients of the latter. In other words, in the first equation of the preceding slide, the LHS and RHS represents the leading term of a coefficient of the holonomy along a tiny square in* U*.*

Review for last lecture

Proof of Gauss theorem:

- $\partial_i x_j = \sum_k \Gamma_{ij}^k x_k + L_{ij} G$, L_{ij} is the i,j-th entry of the matrix representation of second fundamental form.
- $\partial_i \partial_j x_1 = \partial_j \partial_i x_1$
- $EK = \Gamma_{11}^1 \Gamma_{21}^2 + \partial_2 \Gamma_{11}^2 + \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{21}^1 \Gamma_{11}^2 - \partial_1 \Gamma_{21}^2 - \Gamma_{21}^2 \Gamma_{12}^2$

Remark on HW: Two equivalent ways to define induced map between surfaces. $F : S_1 \to S_2$, $F(p) = q$, $DF : T_p(S_1) \to T_q(S_2)$.

- $i : U \to S_1$, $j : V \to S_2$ are two coordinate charts containing p and q respectively, $G = j^{-1} \circ F \circ i$. Then $DF = J(j)J(G)J(i)^{-1}$.
- α is any smooth regular curve on S_1, $\alpha(0) = p$. $DF(\alpha'(0)) = (F \circ \alpha)'(0)$.

Geodesics

Let S be a smooth regular surface, γ a smooth regular space curve under arc length parameterization that lies on S.

Definition 5.3.1

γ is called a **geodesic** on S iff $\gamma'(t) = P_{\gamma,t}\gamma'(0)$.

It follows from the definition that

Theorem 5.3.2

Suppose $\gamma : I \to S$ is a smooth regular curve, $0 \in I$. The followings are equivalent:

1. *γ is a geodesic under arc length parameterization.*
2. *γ is under arc length parameterization, and the curvature of γ equals the absolute value of its normal curvature.*
3. *For any coordinate chart $i : U \to S$, $\beta = i^{-1} \circ \gamma$ satisfies*
   - $E\beta_1'^2 + 2F\beta_1'\beta_2' + G\beta_2'^2 = 1$
   - $\beta_i'' = -\sum_{j,k} \Gamma_{jk}^i \beta_j' \beta_k'$.

From the third point above and Picard's theorem, we have:

Corollary 5.3.3

Let p be any point on a smooth regular surface S, $v \in T_p(S)$ any unit vector, then there is a geodesic γ such that $\gamma(0) = p$, $\gamma'(0) = v$.

Example 5.3.4

- Straight lines are geodesics.
- Great circles are geodesics on spheres.

Proof of Theorem 5.3.2.

- 1 $\implies$ 2: $\gamma'' = \frac{d}{dt}\gamma' = \frac{d}{dt}P_{\gamma,t}(\gamma'(0)) \in N_{\gamma(t)}(S)$, hence the unit normal vector is parallel to γ'', the absolute value of normal curvature equals curvature.
- 2 $\implies$ 3: The first equation is equivalent to $\|\gamma'\| = 1$. Because the curvature equals the absolute value of normal curvature, γ'' is parallel to $G(\gamma(t))$, hence

$$\gamma'' = (i \circ \beta)'' = \frac{d}{dt}(\beta_1' x_1 + \beta_2' x_2)$$

$$= \sum_i (\beta_i'' x_i + \beta_i \sum_{j,k} \Gamma_{ik}^j \beta_k' x_j + \sum_k \beta_k' L_{ik} G)$$

Compare coefficients, we get the second equation.

Proof.

- 3 $\implies$ 1: The first equation implies that γ is under arc length parameterization. Consider $H(t) = \gamma'(t) - P_{\gamma,t}\gamma'(0)$. $H(0) = 0$,

$$H'(t) = \sum_i \beta_i'' x_i + \sum_i \beta_i' (\sum_{j,k} \Gamma_{ik}^j \beta_k' x_j + \sum_k \beta_k' L_{ik} G) + \frac{d}{dt} P_{\gamma,t}\gamma'(0)$$

  Which is in $N_{\gamma(t)}(S)$. $H \cdot x_1$, $H \cdot x_2$ satisfies a homogenuous ODE with zero initial conditions, hence the solution must be 0.

□

Remark

A geodesic is the curve on S whose length is locally minimized. To see this, let γ be a geodesic, a a compactly supported real valued function on I, $\beta_s = \gamma(t) + sa(t)(G(\gamma(t)) \times \gamma'(t))$, then because γ'' is parallel to G,

$$\|\beta_s'\| = \|\gamma' + sa'(G(\gamma(t)) \times \gamma'(t)) + sa((G \circ \gamma)' \times \gamma') + saG \times \gamma''\| \geq 1$$

6 Gauss-Bonnet Theorem, Global differential geometry

- Geodesic curvature
- Calculation in orthogonal coordinate charts
- Gauss-Bonnet Theorem
- Topics that will not be in the final exam

Geodesic Curvature

Definition 6.1.1

Let γ be a smooth regular curve on smooth regular surface S, G is the Gauss map. Then

- Normal curvature $k_n = \gamma'' \cdot G$
- Geodesic curvature $k_g = \gamma'' \cdot (G \times \gamma')$.

It is easy to see that $k^2 = k_n^2 + k_g^2$. Geodesic curvature is a generalization of the concept of signed curvature:

Example 6.1.2

Let $S = \{(x, y, 0)\}$, $G = (0, 0, 1)$, then the geodesic curvature is the signed curvature.

Example 6.1.3

Geodesic curvature of circles on sphere.

Calculation

Let $i : U \to S$ be a coordinate chart, $\beta = i^{-1} \circ \gamma$. Suppose $E{\beta_1'}^2 + 2F\beta_1'\beta_2' + G{\beta_2'}^2 = 1$, i.e. it is under arc length parameterization. Then the orthogonal projection of γ'' on $T_{\gamma(t)}(S)$ is

$$P_{T_{\gamma(t)}}(\gamma'') = \sum_i (\beta_i'' + \sum_{j,k} \Gamma^i_{jk}\beta_j'\beta_k')x_i$$

The absolute value of the geodesic curvature is the squared root of the first fundamental form of this vector. Alternatively, we also have

$$k_g = \gamma' \times P_{T_{\gamma(t)}}(\gamma'') \cdot G(\gamma(t))$$

Example 6.2.1

Suppose $F = 0$, i.e. we are in the case of orthogonal coordinate charts. Find formula for Christoffel symbols, Gaussian curvature and geodesic curvatures.

- Christoffel symbols of first kind:

$$\Gamma_{111} = \frac{1}{2}E_u, \Gamma_{112} = \Gamma_{121} = \frac{1}{2}E_v, \Gamma_{122} = -\frac{1}{2}G_u$$

$$\Gamma_{211} = -\frac{1}{2}E_v, \Gamma_{212} = \Gamma_{221} = \frac{1}{2}G_u, \Gamma_{222} = \frac{1}{2}G_v$$

- Christoffel symbols of second kind:

$$\Gamma^1_{11} = \frac{E_u}{2E}, \Gamma^1_{12} = \Gamma^1_{21} = \frac{E_v}{2E}, \Gamma^1_{22} = -\frac{G_u}{2E}$$

$$\Gamma^2_{11} = -\frac{E_v}{2G}, \Gamma^2_{12} = \Gamma^2_{21} = \frac{G_u}{2G}, \Gamma^2_{22} = \frac{G_v}{2G}$$

Gaussian curvature

$$\frac{1}{E}((\Gamma^2_{11})_v - (\Gamma^2_{21})_u + \Gamma^1_{11}\Gamma^2_{21} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{21}\Gamma^2_{11} - \Gamma^2_{21}\Gamma^2_{12})$$
$$= \frac{1}{E}(-(\frac{E_v}{2G})_v - (\frac{G_u}{2G})_u + \frac{E_u}{2E}\frac{G_u}{2G} - \frac{E_v}{2G}\frac{G_v}{2G} + \frac{E_v}{2E}\frac{E_v}{2G} - \frac{G_u}{2G}\frac{G_u}{2G})$$
$$= -\frac{1}{2\sqrt{EG}}((\frac{E_v}{\sqrt{EG}})_v + (\frac{G_u}{\sqrt{EG}})_u)$$

Geodesic curvature

Let $\beta = i^{-1} \circ \gamma$.

- Approach 1:

$$k_g^2 = E(\beta_1'' + \frac{E_u}{2E}\beta_1'^2 + \frac{E_v}{E}\beta_1'\beta_2' - \frac{G_u}{2E}\beta_2'^2)^2$$

$$+G(\beta_2'' - \frac{E_v}{2G}\beta_1'^2 + \frac{G_u}{G}\beta_1'\beta_2' + \frac{G_v}{2G}\beta_2'^2)^2$$

- Approach 2:

$$k_g = \sqrt{EG}\beta_1'(\beta_2'' - \frac{E_v}{2G}\beta_1'^2 + \frac{G_u}{G}\beta_1'\beta_2' + \frac{G_v}{2G}\beta_2'^2)$$

$$-\sqrt{EG}\beta_2'(\beta_1'' + \frac{E_u}{2E}\beta_1'^2 + \frac{E_v}{E}\beta_1'\beta_2' - \frac{G_u}{2E}\beta_2'^2)$$

Approach 3: Use parallel transport

- For any t, identify $T_{\gamma(t)}(S)$ with $\mathbb{R}^2$ by $D_t : (a, b) \mapsto \frac{ax_1}{\sqrt{E}} + \frac{bx_2}{\sqrt{G}}$.
- Under this identification, Euclidean distance is sent to first fundamental form, and standard basis is sent to a basis with the same orientation as $\{x_1, x_2\}$.
- As a consequence, $D_t^{-1} \circ P_{\gamma,t} \circ D_0$ is an orientation and distance preserving linear transformation from $\mathbb{R}^2$ to itself hence must be rotation with rotation angle θ.
- When $t << 1$, $\theta << 1$, and by definition of parallel transport, we have

$$\theta = \frac{t}{2\sqrt{EG}}(E_v\beta_1' - G_u\beta_2') + o(t)$$

Remark

When γ is a closed curve, $\gamma(1) = \gamma(0)$, by Green's theorem we have the total turning angle when parallel transported through γ (a.k.a. holonomy) is

$$\int_0^1 \frac{dt}{2\sqrt{EG}}(E_v\beta_1' - G_u\beta_2') = \int_\beta \frac{1}{2\sqrt{EG}}(E_v du - G_u dv)$$

$$= \int_{\text{Region bounded by } \beta} -(\frac{E_v}{2\sqrt{EG}})_v - (\frac{G_u}{2\sqrt{EG}})_u dudv$$

$$= \int_{\text{Region bounded by } \gamma} \text{Gaussian Curvature } d \text{ Area}$$

This made precise the argument on Slide 141.

Review for last lecture

- Geodesic curvature $k_g = \gamma'' \cdot (G \times \gamma') = G \cdot (\gamma' \times \gamma'')$
- Calculation of geodesic curvature: use the fact that

$$P_{T_{\gamma(t)}}(\gamma'') = \sum_i (\beta_i'' + \sum_{j,k} \Gamma^i_{jk} \beta_j' \beta_k') x_i$$

- Gaussian curvature under orthogonal coordinate chart:

$$-\frac{1}{2\sqrt{EG}}((\frac{E_v}{\sqrt{EG}})_v + (\frac{G_u}{\sqrt{EG}})_u)$$

- Combining with Green's theorem:

Theorem 6.2.2

The holonomy along γ is $\int_\Omega KdA$ where A is the area form on S.

Existence of orthogonal coordinate charts

Theorem 6.2.3

Let S be a smooth regular surface, then around any point $p \in S$ there is an orthogonal coordinate chart.

Proof idea

Find a nearby point q, make "polar coordinate" centered at q.

Geodesic curvature under orthogonal coordinate charts

Assume that γ is under arc length parameterization,

Theorem 6.2.4

$$(\frac{d}{dt}(P_{\gamma,t} \circ P_{\gamma,t_0}^{-1}(\gamma'(t_0)) \times \gamma'(t))) \cdot G(\gamma(t_0))|_{t=t_0} = k_g(t_0)$$

Proof.

$$(\frac{d}{dt}(P_{\gamma,t} \circ P_{\gamma,t_0}^{-1}(\gamma'(t_0)) \times \gamma'(t)))|_{t_0} \cdot G = (CG \times \gamma'(t_0) + \gamma'(t_0) \times \gamma''(t_0)) \cdot G$$

$$= (\gamma' \times \gamma'') \cdot G|_{t_0} = \gamma'' \cdot (G \times \gamma')|_{t_0} = k_g(t_0)$$

□

As a consequence:

Theorem 6.2.5

Under orthogonal coordinate, when $\beta = i^{-1}\gamma$, let ϕ be the angle between γ' and x_u, then

$$k_g = \phi' + \frac{1}{2\sqrt{EG}}(-E_v\beta_1' + G_u\beta_2')$$

Local Gauss-Bonnet Theorem

Theorem 6.3.1 (Hopf theorem)

If γ is a smooth simple closed curve bounding a small region within an orthogonal coordinate chart, and travel around it counterclockwise when seen from the direction of the Gauss map, then $\int_0^T \phi' dt = 2\pi$, here $T > 0$ is the smallest positive number such that $\gamma(T) = \gamma(0)$.

As a consequence,

Theorem 6.3.2

Let γ be a smooth simple closed curve on S bounding a small enough region Ω, and travel around Ω counterclockwise when seen from the direction of the Gauss map. Then $\int_\gamma k_g ds + \int_\Omega K dA = 2\pi$.

Definition 6.3.3

We call a curve γ a piecewise smooth curve, if its domain can be decomposed into finitely many intervals, and γ is smooth on the closure of each. Let t_0 be a boundary point of two consecutive intervals, the angle from $\lim_{h\to 0-}\frac{\gamma(t_0+h)-\gamma(t_0)}{h}$ to $\lim_{h\to 0+}\frac{\gamma(t_0+h)-\gamma(t_0)}{h}$ is called the turning angle at t_0.

Because piecewise smooth curves can be approximated by smooth curves, we have:

Theorem 6.3.4 (Local Gauss-Bonnet)

Let γ be a piecewise smooth simple closed curve on S bounding a small enough region Ω, and travel around Ω counterclockwise when seen from the direction of the Gauss map. Let θ_k be all the turning angles, then $\sum_k \theta_k + \int_\gamma k_g ds + \int_\Omega K dA = 2\pi$.

Review: Outline for the argument to Local Gauss-Bonnet

Let γ be a smooth closed curve on S bounding a region Ω which is on its left and is contained in a coordinate neighborhood.

1. Around every point there is an orthogonal coordinate chart.
2. Gaussian curvature under orthogonal coordinate chart:

$$-\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

3. Identify $T_{\gamma(t)}(S)$ with $\mathbb{R}^2$ by $(a,b)\mapsto \frac{a}{\sqrt{E}}x_1+\frac{b}{\sqrt{G}}x_2$.
4. Parallel transport is a rotation by angle $\theta(t)$, such that

$$\theta'=\frac{1}{2\sqrt{EG}}(E_v\beta_1'-G_u\beta_2')$$

5. θ is the angle from x_1 to $P_{\gamma,t}(x_1)$, viewed from the direction of G.

6. Let θ_1 be the angle from $P_{\gamma,t}\gamma'(0)$ to $\gamma'(t)$. Then

$$\theta_1'(s) = (\frac{d}{dt}(P_{\gamma,t} \circ P_{\gamma,s}^{-1}(\gamma'(s)) \times \gamma'(t)))|_{t=t_0} \cdot G(\gamma(t_0)) = k_g(t_0)$$

6. $\theta_1' = \phi' - \theta'$, where ϕ is the angle from x_1 to γ'.
6. By Green's theorem,

$$\int_\gamma k_g ds + \int_\Omega K dA = \int_\gamma \phi' ds = 2\pi$$

The last step is by Hopf's theorem.

Global Gauss-Bonnet

Definition 6.3.5

If M is a finite cell complex (finitely many discs of different dimensions glued together, discs of higher dimension has boundary in discs of lower dimension), then the Euler characteristic is the number of even dimensional cells minus the number of odd dimensional cells, called $\chi(M)$

Example 6.3.6

A single disc has $\chi = 1$.

One can show that any compact region on S bounded by piecewise smooth curves can be decomposed into small regions bounded by a single piecewise smooth closed curve, and that when one glue two regions along the boundary, integral of k_g and K are additive, when the gluing is via m intervals, Euler characteristic is subtracted by m and the turning angle is subtracted by $2m\pi$. The gluing can also happen along cylinders, during which the Euler characteristics, turning angles and integral of k_g and K are all unchanged. Hence we have

Theorem 6.3.7 (Gauss-Bonnet)

Let Ω be a compact region on smooth regular oriented surface S bounded by piecewise smooth curves which travel counterclockwise around Ω, with turning angles θ_k, then

$$\sum_k \theta_k + \int_\gamma k_g ds + \int_\Omega K dA = 2\pi\chi(\Omega)$$

Exponential map and geodesic polar coordinate

Definition 6.4.1

The Exponential map is defined as $\exp : U \to S$, $exp(v) = \gamma_v(1)$, where U is a neighborhood of 0 in $T_p(S)$, $\gamma_v(\|v\|t)$ is a geodesic under arc length parameterization, such that $\gamma(0) = p$, $\gamma'(0)$ is in the direction of v.

Definition 6.4.2

The geodesic polar coordinate is the composition of exp and $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$.

Theorem 6.4.3

The exponential map when restricted to a small enough neighborhood of 0*, is a coordinate chart.*

The proof is by the continuous dependence on initial conditions of ODE.

Theorem 6.4.4

Under the geodesic polar coordinate, $F = 0$, $E = 1$, $\sqrt{G}_{rr}/\sqrt{G} = -K$.

Proof.

$$F_r = \partial_r x_r \cdot x_\theta + x_r \cdot \partial_r x_\theta = \frac{1}{2}\partial_\theta(x_r \cdot x_r) = 0$$

Obviously $E = 1$, hence

$$K = -\frac{1}{2\sqrt{EG}}((\frac{E_\theta}{\sqrt{EG}})_\theta + (\frac{G_r}{\sqrt{EG}})_r) = -\frac{1}{2\sqrt{G}}(\frac{G_r}{\sqrt{G}})_r = -\sqrt{G}_{rr}/\sqrt{G}$$

□

Remark

It is easy to see that $G = 0$ *when* $r = 0$. *Hence two surfaces with the same constant Gaussian curvatures are locally isometric.*

Proof of Hopf's theorem

1. The integral must be $2k\pi$, hence unchanged under continuous deformation.
2. Deform E and G to 1.
3. Find the point where v is smallest, reparametrized so that this point is $\gamma(0)$.
4. Replace tangent with secant line.
5. Deform the end points of the secant line.

Summary of the philosophy of local differential geometry

- Differential geometry of a manifold is represented via connections, i.e., parallel translations, on its tangent bundles (or other bundles).
- Holonomy of the connection measures the "intrinsic curvature".
- Difference between the connection of a sub manifold and the larger space (the L_{ij}, k, τ etc) measures the "extrinsic curvature".

What can we go from here

- Further topics of global differential geometry: Jacobi field and comparison theorem, 4 vertex theorem, classification of surfaces with zero curvature, rigidity of sphere etc.
- Generalizations to differential geometry on general bundles.
- Generalizations of Gauss-Bonnet: Chern-Weil theory, Hodge theory, K-theory, index theory.
- Connection with PDE: Geometric Analysis.
- Connection with physics: Gauge theory, Yang-Mills equation etc.
- Symplectic, complex and algebraic geometry.

7 Final Review

- Differential geometry of plane curves and space curves
  - Arc length function and arc length parameterization
  - Signed curvature, curvature and torsion
  - Structural equation. Fundamental theorems.
- Basic concepts of smooth regular surfaces
  - Coordinate charts via implicit function theorem, change of coordinates, smooth functions, smooth maps.
  - Tangent space and normal space, induced maps on tangent space.
  - First fundamental form
  - Orientation and Gauss map, shape operator, second fundamental form, principal, mean and Gaussian curvature.

- Gauss's theorem and Gauss-Bonnet
  - Parallel transport and Christoffel symbols, Christoffel symbols are intrinsic.
  - Gauss's theorem.
  - Holonomy along a closed curve.
  - Local and global Gauss-Bonnet.
- Curves on surfaces
  - Normal curvature and geodesic curvature.
  - Geodesics and geodesic equation.

Key idea:

- Geometry is all about how to move a vector around in a vector bundle (parallel transport for surfaces, unit tangent, normal, binormal vectors for curves)
- How much the vectors are rotated under such movement characterizes extrinsic curvature (shape operator for surfaces, curvature and torsion for curves)
- Holonomy characterizes intrinsic curvature.

Example 7.0.1

Consider the surface $S = \{(x, y, z) : (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\}$.

- Show that S is a smooth regular surface.
- Show that when $u \in (-\pi, \pi)$, $v \in (-\pi, \pi)$, $f : (u, v) \mapsto (2\cos u + \cos u \cos v, 2\sin u + \sin u \cos v, \sin v)$ is a coordinate chart.
- Find the first fundamental form, shape operator, second fundamental form, principal curvatures, Christoffel symbols on S.
- Verify that the integral of the Gaussian curvature on S is 0.

Answer:

- This is because the function $(\sqrt{x^2 + y^2} - 2)^2 + z^2$ is regular on $\{(x, y, z) : x^2 + y^2 \neq 0,$ and either $z \neq 0$ or $x^2 + y^2 \neq 4\}$.
- Check that it is a homeomorphism to an open subset of S, smooth, and the Jacobian has rank 2.

- Tangent space at $f(u, v)$ is spanned by

$$x_1 = (-2\sin u - \sin u \cos v, 2\cos u + \cos u \cos v, 0)$$

and

$$x_2 = (-\cos u \sin v, -\sin u \sin v, \cos v)$$

So $E = x_1 \cdot x_1 = (2+\cos v)^2,\ F = 0,\ G = 1.$ Now calculate the Gauss map:

$$G \circ f = \frac{x_1 \times x_2}{\|x_1 \times x_2\|}$$

$$= (\cos u \cos v, \cos v \sin u, \sin v)$$

$$J(G \circ f) = \begin{bmatrix} -\sin u \cos v & -\cos u \sin v \\ \cos u \cos v & -\sin u \sin v \\ 0 & \cos v \end{bmatrix}$$

- $$(J(f)^T J(f))^{-1} = \begin{bmatrix} (2+\cos v)^{-2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$J(f)^T = \begin{bmatrix} -2\sin u - \sin u \cos v & 2\cos u + \cos u \cos v & 0 \\ -\cos u \sin v & -\sin u \sin v & \cos v \end{bmatrix}$$

So second fundamental form is

$$-J(f)^T J(G \circ f) = \begin{bmatrix} -\cos v(2+\cos v) & 0 \\ 0 & -1 \end{bmatrix}$$

The matrix representation of shape operator is

$$(J(f)^T J(f))^{-1} J(f)^T J(G \circ f) = \begin{bmatrix} \frac{\cos v}{2+\cos v} & 0 \\ 0 & 1 \end{bmatrix}$$

Principal curvatures are $-\frac{\cos v}{2+\cos v}$, -1, Gaussian curvature is $\frac{\cos v}{2+\cos v}$.

- Non-zero Christoffel symbols of first kind:

$$\Gamma_{211} = \sin v(2 + \cos v), \Gamma_{121} = \Gamma_{112} = -\sin v(2 + \cos v)$$

Non-zero Christoffel symbols of second kind:

$$\Gamma^2_{11} = \sin v(2 + \cos v), \Gamma^1_{12} = \Gamma^1_{21} = -\frac{\sin v}{2 + \cos v}$$

Remark

One can verify that the Christoffel symbols calculated above satisfies their definitions. Also, the Gaussian curvature can be calculated using formula

$$K = -\frac{\sqrt{E}_{22}}{\sqrt{E}}$$

- Now do integration

$$\int_{-\pi}^{\pi} du \int_{-\pi}^{\pi} dv (2 + \cos v) \cdot \frac{\cos v}{2 + \cos v} = 0$$

Example 7.0.2

Let S be a surface with first fundamental form under a coordinate chart f being $E = 1,\ G = e^u,\ F = 0$.

- Find Christoffel symbols.
- Find the geodesic curvature of the curve $\gamma : t \mapsto f(1, t),\ t \in [-1, 1]$.
- Find the parallel transport of x_1 along the curve above to $\gamma(1)$.

- Non-zero Christoffel symbols of first kind:

$$\Gamma_{122} = -e^u/2, \Gamma_{212} = \Gamma_{221} = e^u/2$$

Non-zero Christoffel symbols of second kind:

$$\Gamma^1_{22} = -e^u/2, \Gamma^2_{12} = \Gamma^2_{21} = 1/2$$

- Arc length reparameterization:

$$\alpha = f^{-1} \circ \gamma : t \mapsto (1, t)$$

Arc length function, $t_0 = 0$:

$$s(t) = \int_0^t e^{1/2} dr = e^{1/2} t$$

So,

$$\beta = f^{-1} \circ \gamma \circ s^{-1} : s \mapsto (1, s/e^{1/2})$$

$$P_{T_{(\gamma \circ s^{-1})^{-1}(t)}}((\gamma \circ s^{-1})'') = -\frac{x_1}{2}$$

$$(\gamma \circ s^{-1})' = x_2/e^{1/2}$$

$$G \circ \gamma \circ s^{-1} = \frac{x_1 \times x_2}{e^{1/2}}$$

$$k_g = (\gamma \circ s^{-1})'' \cdot (G \times (\gamma \circ s^{-1})') = P_{T_{(\gamma \circ s^{-1})^{-1}(t)}}((\gamma \circ s^{-1})'') \cdot (G \times (\gamma \circ s^{-1})')$$

$$= -\frac{x_1}{2} \cdot \left(\frac{x_1 \times x_2}{e^{1/2}} \times \frac{x_2}{e^{1/2}}\right) = \frac{x_1 \times x_2}{e^{1/2}} \cdot \frac{x_1 \times x_2}{2e^{1/2}} = \frac{1}{2}$$

- First way to find the parallel transport: the turning angle from $P_{(\gamma\circ s^{-1}),t}(\gamma\circ s^{-1})'$ to $(\gamma\circ s^{-1})'$ changes with the speed k_g under arc length parameterization. So this angle at time $t=1$ becomes $\int_0^{e^{1/2}}\frac{1}{2}ds=\frac{e^{1/2}}{2}$, x_1 will be sent to $\cos(-\frac{e^{1/2}}{2})x_1+\sin(-\frac{e^{1/2}}{2})\frac{x_2}{e^{1/2}}$.
- Second way to find the parallel transport: via the differential equation. Suppose $P_{\gamma,t}(x_1)=a(t)x_1+b(t)x_2$. Then

$$0=P_{T_{\gamma(t)}(S)}\frac{d}{dt}P_{\gamma,t}(x_1)=a'(t)x_1+\frac{a(t)}{2}x_2+b'(t)x_2-\frac{b(t)e}{2}x_1$$

So

$$a'=eb/2, b'=-a/2, a(0)=1, b(0)=0$$

$$a(t)=\cos(\frac{e^{1/2}}{2}t), b(t)=-\frac{\sin(\frac{e^{1/2}}{2}t)}{e^{1/2}}$$

Remark

Two ways to calculate parallel transport:

- *Use k_g: the angle from $P_{\gamma,t}(\gamma'(0))$ to $\gamma'(t)$ changes at speed k_g.* **Need to make sure γ is arc-length parameterization.**
- *Use definition: $P_{\gamma,t}(v) = a_1(t)x_1 + a_2(t)x_2$*

$$0 = P_{T_{\gamma(t)}(S)} \frac{d}{dt} P_{\gamma,t}(v) = \sum_i (a_i'(t) + \sum_{j,k} \Gamma^i_{jk} a_j \beta_k') x_i$$

So

$$a_i'(t) + \sum_{j,k} \Gamma^i_{jk} a_j \beta_k' = 0$$

Here $\beta = f^{-1} \circ \gamma$, where f is the coordinate chart. **Do not need to make sure that γ is arc-length parameterization.**

Example 7.0.3

If three smooth oriented regular surfaces M, N, R intersects at only a single point p, show that:

- Their tangent spaces at p coincides.
- Choose orientation such that all their Gauss maps are identical at p. If N lies between M and R, M and R all have positive mean and Gaussian curvature, then so is N.

Example 7.0.4

If S is a smooth oriented surface with non-positive curvature. α, β are two geodesic segments on S intersecting at two end points. Use Gauss-Bonnet, show that α and β can not bound a region that is homeomorphic to a disk.

Example 7.0.5

If S and T are two smooth oriented surfaces intersecting on a simple closed smooth curve γ, and at any point on γ S and T has the same tangent space. γ bounds a disk D_1 on S and another disk D_2 on T. S and T both have positive Gaussian curvature. Show that the image of D_1 and D_2 under Gauss map have either the same area or the areas add up to 4π.

Example 7.0.6

Let S be a smooth oriented surface, γ a geodesic passing through a point p, β another smooth regular curve intersects with γ only at p, and seen from the direction of the Gauss map, β is to the right of γ. Show that β is tangent to γ at p, and if their tangent vectors are at the same direction at p, the geodesic curvature of β is non-positive at p.

- There will not be complicated computation or proofs. Should be similar or easier than HW and practice problems.
- Don't panic if you can not do a specific problem! Just move on to the next.
- Letter grade range provided in the beginning of the semester is just the minimum. I may adjust the grades upwards. Average should be at around AB.
- Don't forget to do the course evaluation if you haven't done it already!
- Good luck with the exams and have a nice summer break!