Math 521 Analysis I

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Course Description: This is an introductory course on mathematical analysis. We will cover basic ideas in analysis like real numbers, limits, continuity, differentiation and integration. It is a more rigorous, proof based treatment of what we have seen in Calculus classes.

How to succeed in this course:

I know that we all have different academic backgrounds and different learning styles, so what works for one may not work for another. However the following is what I found helpful when I learned this subject myself:

1. Make sure that one understands every definition and every step of every proof.
2. Figure out the key idea behind every concept or theorem, and summarize it in one sentence.
3. Do as many exercises as possible. To save time it is often not necessary to write down a complete solution, just figure out the key idea then move on.

Rudin's book is one of the most standard introductory text on analysis and (in my opinion) one of the more accessible one. If you finished it maybe check out his *Real and Complex Analysis* or Jean Dieudonné's *Foundations of Modern Analysis* which cover more advanced topics. There are also textbooks, like the one by Grigorii Fichtenholz, that are at the same level of Rudin's Principles of Mathematical Analysis but may contain more examples and application.

Contents

1 Introduction to Naive Set Theory

A thorough treatment of logic and axiomatic set theory may take a whole semester. So we will only mention some concepts, notations and results from set theory that are widely used in analysis.

1.1 Set, subsets, specification and power set

Remark 1.1.1. Let A be a **set**. We write $x \in A$ if x is an **element**, or a **member** of A.

Example 1.1.2. $2 \in \mathbb{N}$, $1.5 \notin \mathbb{Z}$.

Definition 1.1.3. We say a set B is a **subset** of a set A, if for every $x \in B$, $x \in A$. We denote it as $B \subseteq A$. If there is at least one element in A which does not belong to B, we call B a **proper subset**, denoted as $B \subsetneq A$.

Remark 1.1.4. Two sets are considered **equal** if they contain the same elements. In other words, $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Definition 1.1.5. A set that contains no element is called an **empty set**, denoted as $\emptyset$.

Remark 1.1.6. The empty set is unique by definition. Here by "unique" we mean that if A and B are both empty sets, then $A = B$. Also note that $\emptyset \neq \{\emptyset\}$, Because $\{\emptyset\}$ has a single element which is $\emptyset$. $\emptyset$ is a subset of every set.

Remark 1.1.7. To describe a set, one can write down its elements and put a $\{\}$ around them, for example:

$$A = \{0, 1, 2\}$$

One can also use the terminology of **specification** (or **comprehension**) to describe a set consisting of elements that satisfy a certain property. For example:

$$A = \{n \in \mathbb{N} : n \leq 2\}$$

means "A is a set consists of natural numbers that are no more than 2"

$$B = \{\sin(n) : n \in Z\}$$

means "B is a set consisting of the sines of integers".

Remark 1.1.8. The ":" above can be "|" in some texts.

Remark 1.1.9. Note that in both examples in Remark 1.1.7 above, the variable n is required to be a member of some known set, like $\mathbb{N}$ or $\mathbb{Z}$, i. e. we are doing **restricted specification**. The result of **unrestricted specification** may not necessarily be a set, for example $\{x : x \notin x\}$ can not be a set due to Russell's paradox.

Definition 1.1.10. If A is a set, the collection of all its subsets also forms a set, denoted as $P(A)$ or 2^A, called the **power set**.

Remark 1.1.11. Power sets are never empty because $\emptyset \in P(A)$ for all set A. If A has n elements, the number of elements in $P(A)$ is 2^n.

Remark 1.1.12. Notations for some common sets of numbers:

- $\mathbb{N}$: set of natural numbers. In mathematics we usually think of 0 as a natural number.
- $\mathbb{Z}$: set of integers
- $\mathbb{R}$: set of real numbers. This is a key concept for this semester which we will discuss in more details later.
- $\mathbb{C}$: set of complex numbers.

Example 1.1.13. $\{B \in P(\mathbb{N}) : 1 \in B\}$ is the set of all subsets of the set of natural numbers that contains 1.

1.2 Unions, intersections and set differences

Definition 1.2.1. Let A be a set of sets. The **union** of elements in A, denoted as $\bigcup A$, or $\bigcup_{B\in A} B$, or (when $A = \{B_\alpha : \alpha \in I\}$) $\bigcup_{\alpha\in I} B_\alpha$, is a set that satisfies: $x \in \bigcup A$ iff there is some $B \in A$, such that $x \in B$.

Definition 1.2.2. Let A and B be two sets, $A \cup B$ is a set such that $x \in A \cup B$ iff $x \in A$ or $x \in B$.

Definition 1.2.3. Let A be a non-empty set of sets, then the **intersection** $\bigcap A$ or $\bigcap_{B\in A} B$ or (when $A = \{B_\alpha : \alpha \in I\}$) $\bigcap_{\alpha\in I} B_\alpha$, is defined as

$$\bigcap A = \{x \in \bigcup A : x \in B \text{ for all } B \in A\}$$

Definition 1.2.4. Let A and B be two sets, $A \cap B$ is defined as $\{x \in A \cup B : x \in A, x \in B\}$.

Definition 1.2.5. Let A and B be two sets, $A\backslash B$ ($A - B$ in some texts) is defined as $A\backslash B = \{a \in A : a \notin B\}$.

Example 1.2.6.

- $A \cup A = A \cap A = A$, $A\backslash A = \emptyset$.
- $(1, 2] \cup (3/2, 3] = (1, 3]$. Here $(a, b]$ means $\{x \in \mathbb{R} : a < x \leq b\}$.
- For any natural number n, let $A_n = [n, \infty) = \{x \in \mathbb{R} : x \geq n\}$. Then $\bigcup_{n\in\mathbb{N}} A_n = [0, \infty)$, $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$.

1.3 Cartesian product and relations

Definition 1.3.1. Let A and B be two sets (can be equal). The **Cartesian product** between A and B is the set consisting of all **ordered pairs** of elements in A and elements in B. We can write this as

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Definition 1.3.2. A subset of $A \times B$ is called a **relation** between A and B.

Example 1.3.3.

- If A or B is empty, then $A \times B = \emptyset$.
- $\emptyset$ and $A \times B$ are both relations between A and B.
- If B is non-empty, let $b \in B$, then $\{(a, b) : a \in A\}$ is a relation between A and B.

1.4 Functions, orders and equivalent relations

We are going to see three families of relations for this course:

1.4.1 Functions

Definition 1.4.1. A **function** or a **map** f between A and B is a relation between A and B, such that for any $a \in A$, there is a unique $b \in B$, such that $(a, b) \in f$. We denote this b as $f(a)$. We often write such a function as

$$f : A \to B, a \mapsto f(a)$$

Definition 1.4.2. The set A is called the **domain**, B is called the **codomain**, the set (which is a subset of B) $\{f(a) : a \in A\}$ is called the **range**.

Definition 1.4.3. A function f is called an **injection** if $f(a) = f(b)$ implies $a = b$. It is called a **surjection** if the range equals codomain. It is called a **bijection** if it is both an injection and a surjection.

Example 1.4.4.

- $f : \mathbb{Z} \to \mathbb{Q}$, $n \mapsto n^2/2$ is a function. The domain is $\mathbb{Z}$, codomain is $\mathbb{Q}$, it consists of pairs of the form $(n, n^2/2)$ where n is an integer. It is neither an injection nor a surjection.
- $g : A \to P(A)$, $g(a) = A\backslash\{a\}$, is an injection but not a surjection.
- $h : P(\mathbb{N} \to \mathbb{N}$ defined as

  $$h(A) = \begin{cases} 0 & A = \emptyset \\ \text{smallest number in } A & A \neq \emptyset \end{cases}$$

  is a surjection but not an injection.

Definition 1.4.5. Suppose B is not empty, let $b \in B$, then we can define the **constant function** $c_b : A \to B$ which sends any $a \in A$ to b.

Definition 1.4.6. The **identity function** from A to itself is defined as $id_A : A \to A$, $a \mapsto a$. Or, $id_A = \{(a,a) : a \in A\} \subseteq A \times A$.

Remark 1.4.7. id_A is always a bijection.

Definition 1.4.8. Let $A \subseteq B$, then the **inclusion function** from A to B is defined as $a \mapsto a$.

Remark 1.4.9. An inclusion function is always an injection.

Definition 1.4.10. Let f be a function from A to B, g be a function from B to C.

- The **composition** between f and g, denoted as $g \circ f$, is defined as $a \mapsto g(f(a))$.
- If $C = A$, $g \circ f = id_A$, $f \circ g = id_B$, we call g the **inverse** of f, denoted as $g = f^{-1}$, and f to be **invertible**.

Remark 1.4.11. Identity function composed with any other function equals the other function itself.

Remark 1.4.12. A function is invertible iff it is a bijection.

Remark 1.4.13. Composition of injections is an injection. Composition of surjections is a surjection.

Definition 1.4.14. Let $f : A \to B$ be a function, $A' \subseteq A$, i the inclusion function from A' to A, then the **restriction** of f on A', denoted as $f|_{A'}$, is defined as $f \circ i$.

Definition 1.4.15. Let $f : A \to B$ be a function. Then for every $C \subseteq A$, we define

$$f(C) = \{f(a) : a \in C\}$$

For every $D \subseteq B$, we define

$$f^{-1}(D) = \{a \in A : f(a) \in D\}$$

Remark 1.4.16. Any function is the composition of a surjection with an injection. Let $f : A \to B$ be a function, f can always be seen as the composition of surjection $f : A \to f(A)$ and the inclusion map from the range $f(A)$ of f to B.

Example 1.4.17.

- If $f : \mathbb{N} \to P(\mathbb{N})$ is defined as $f(n) = \mathbb{N} \backslash \{n\}$, $g : P(\mathbb{N}) \to \mathbb{N}$ is defined as

$$g(A) = \begin{cases} 0 & A = \emptyset \\ \text{smallest element in } A & A \neq \emptyset \end{cases}$$

Then

$$(g \circ f)(n) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$$

- If g is the same as above, $f' : \mathbb{N} \to P(\mathbb{N})$ is defined as

$$f'(n) = \{x \in \mathbb{N} : x \geq n\}$$

Then $g \circ f' = id_{\mathbb{N}}$ but $f' \circ g \neq id_{P(\mathbb{N})}$. For example, $f' \circ g(\emptyset) = \mathbb{N}$. Hence f' is not an inverse of g.

- Let $h : P(\mathbb{N}) \to P(\mathbb{N})$ be defined as $P(A) = \mathbb{N} \backslash A$. Then $h \circ h = id_{P(\mathbb{N})}$, $h = h^{-1}$.

Remark 1.4.18. The set of all functions from A to B is denoted as $Map(A, B)$ or B^A.

Example 1.4.19. The composition of functions gives a map

$$\circ : Map(A, B) \times Map(B, C) \to Map(A, C)$$

$$(f, g) \mapsto g \circ f$$

Theorem 1.4.20. There is a bijection between $Map(A \times B, C)$ and $Map(A, Map(B, C))$, defined as

$$f \mapsto (a \mapsto (b \mapsto f(a, b))$$

Proof. It is easy to show that the map from $Map(A \times B, C)$ to $Map(A, Map(B, C))$ defined above is well defined.

We can also verify that the map from $Map(A, Map(B, C))$ to $Map(A \times B, C)$ defined by $g \mapsto ((a, b) \mapsto (g(a))(b))$ is its inverse, so it is a bijection. □

Remark 1.4.21. This is called **Currying** after the American logician Haskell Curry. It is often used in computer science to reduce multi variable functions to single variable ones. For example, the binary operator $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ can be seen as a function sending integer a to a function $b \mapsto a + b$.

Remark 1.4.22. The concept of functions allow us to define Cartesian product of possibly infinite families of sets: Let A be a non-empty set of sets, we can define

$$\prod A = \prod_{B \in A} B = \{f \in Map(A, \bigcup A) : f(B) \in B \text{ for all } B \in A\}$$

When all elements of A are non-empty, the **axiom of choice** (AC) says that this product is non-empty. For this course, as is in most math courses, we always assume AC.

1.4.2 Orders

Definition 1.4.23. Let A be a set. A **partial order** $\preceq$ is a relation from A to itself such that

- For any $a \in A$, $(a, a) \in \preceq$.

- If $(a, b) \in \preceq$, $(b, a) \in \preceq$, then $a = b$.
- If $(a, b) \in \preceq$ and $(b, c) \in \preceq$ then so is (a, c).

If furthermore for any $a, b \in A$, $(a, b) \in \preceq$ or $(b, a) \in \preceq$, we call it a **total** or **linear** order. We often write $(a, b) \in \preceq$ as $a \preceq b$.

Remark 1.4.24. If $a \preceq b$ and $a \neq b$ we write $a \prec b$.

Remark 1.4.25. If $\preceq$ is a partial or linear order, the relation $\preceq'$ defined as $a \preceq' b$ iff $b \preceq a$ is also a partial or linear order respectively.

Example 1.4.26.

- The usual $\leq$ on $\mathbb{N}$ is a linear order.
- $A \preceq B$ iff $B \subseteq A$ is a partial order on $P(\mathbb{N})$ but not a linear order. For example, $\{1\} \not\preceq \{2\}$, $\{2\} \not\preceq \{1\}$.
- $A \preceq' B$ iff there is a bijection from A to B is not a partial order on $P(\mathbb{N})$. This is because $\{1\} \preceq' \{2\}$, $\{2\} \preceq' \{1\}$ but $\{1\} \neq \{2\}$.
- There is a linear order $\preceq$ on $P(\mathbb{N})$ defined as follows:
  - If $A = B$ then $A \preceq B$.
  - If $A \neq B$, let n be the smallest natural number that belongs to either A or B but not the other. If $n \in A$ then $B \preceq A$, if $n \in B$ then $A \preceq B$.

  We can check that it is a linear order as follows:

  - By construction, for any $A \in P(\mathbb{N})$, $A \preceq A$.
  - Suppose $A \neq B$, then there must be some natural number that belongs to either A or B but not the other. Let n be the smallest such number, then if $n \notin A$ then $B \not\preceq A$, if $n \notin B$ then $A \not\preceq B$. Hence $A \preceq B$ and $B \preceq A$ implies $A = B$.
  - The argument above can also be used to show that $A \preceq B$ or $B \preceq A$.
  - Suppose $A \preceq B$, $B \preceq C$. If either $A = B$ or $B = C$ then $A \preceq C$ is obvious. If $A \neq B$, $B \neq C$, let n be the smallest number that belongs to either A or B but not the other, n' be the smallest number that belong to either B or C but not the other, then $n \in B$ and $n' \notin B$, hence $n \neq n'$. Suppose $n < n'$, then $n \in C$, $n \notin A$, and all natural numbers less than n are either in all of A, B and C, or in none of the three. Hence $A \preceq C$. The case when $n > n'$ is analogous.

1.4.3 Equivalence relations

Definition 1.4.27. Let A be a set, an **equivalence relation** on A is a relation $\sim$ between A and A, that satisfies:

- For any $a \in A$, $(a, a) \in \sim$.

- $(a, b) \in\sim$ iff $(b, a) \in\sim$.
- If $(a, b) \in\sim$, $(b, c) \in\sim$, then $(a, c) \in\sim$.

We usually write $(a, b) \in\sim$ as $a \sim b$.

Example 1.4.28.
- $id_A = \{(a, a) : a \in A\}$ is an equivalence relation.
- $A \times A$ is an equivalence relation.
- Let $f : A \to B$ be a map, then $\sim_f$ defined as $a \sim_f b$ iff $f(a) = f(b)$ is an equivalence relation.

Definition 1.4.29. When $\sim$ is an equivalence relation on a set A, the **quotient set** is defined as

$$A/ \sim= \{\{b \in A : b \sim a\} : a \in A\}$$

Here the set $\{b \in A : b \sim a\}$ is called the **equivalence class containing (or represented by)** a, denoted as $[a]$.

Example 1.4.30.
- If $\sim= id_A$, $A/ \sim= \{\{a\} : a \in A\}$
- If $\sim= A \times A$, $A/ \sim= \{A\}$.

1.5 Cardinality

Definition 1.5.1. Let A and B be two sets. If there is a bijection from A to B, we say that they have the same **cardinality**, denoted as $|A| = |B|$.

Definition 1.5.2. If there is an injection from A to B then we write $|A| \leq |B|$. If $|A| \leq |B|$ and $|A| \neq |B|$ we write $|A| < |B|$.

Remark 1.5.3. By AC, if both A and B are non-empty, there is an injection from A to B would be equivalent to there is a surjection from B to A.

Theorem 1.5.4. Let A and B be two sets, then either $|A| \leq |B|$, or $|B| \leq |A|$. Furthermore, if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

The proof of this theorem can be found in textbooks on logic or set theory.

Remark 1.5.5. Let A be any set, define equivalence relation $\sim$ on $P(A)$ as $B \sim C$ iff $|B| = |C|$. Then Theorem 1.5.4 implies that $|\cdot| \leq |\cdot|$ is a linear order in $P(A)/ \sim$.

Example 1.5.6. $|\mathbb{Z}| = |\mathbb{N}|$. A bijection from $\mathbb{Z}$ to $\mathbb{N}$ can be defined as, for example,

$$f(n) = \begin{cases} 2n & n \geq 0 \\ -2n - 1 & n < 0 \end{cases}$$

There isn't a set with the largest possible cardinality, because of the following theorem:

Theorem 1.5.7. (Cantor's **diagonal argument**) Let A be a set, then $|A| < |P(A)|$.

Proof. The map $a \mapsto \{a\}$ is an injection from A to $P(A)$, hence $|A| \leq |P(A)|$. Now we only need to prove that $|A| \neq |P(A)|$. Suppose there is a bijection $f : A \to P(A)$. Let $B = \{a \in A : a \notin f(a)\}$. Because f is surjective, there must be some $b \in A$ such that $B = f(b)$. If $b \in B = f(b)$ by construction of B, $b \notin B$. If $b \notin B = f(b)$, by construction of B, $b \in B$. Both possibilities result in contradiction. Hence such a bijection f can not exist. $\square$

Example 1.5.8. $|P(\mathbb{N})| > |\mathbb{Z}|$.

1.5.1 Finite sets and infinite sets

Theorem 1.5.9. Let A be a set. The followings are equivalent:

1. There is a bijection between A and a proper subset of A.
2. There is an injection from $\mathbb{N}$ to A.
3. There isn't a bijection from A to $\{x \in \mathbb{N} : x < n\}$.

The proof is unrelated to analysis but we include it here as an illustration of the mathematical induction.

Proof. **1 implies 3**: Let f be a bijection from A to some $B \subsetneq A$. Suppose $g : A \to \{x \in \mathbb{N} : x < n\}$ is a bijection, then $g \circ f \circ g^{-1}$ is an injection from $\{x \in \mathbb{N} : x < n\}$ to itself which is not a bijection. One can show that such an injection can not exist by the "pigeon hole principle", or by mathematical induction as follows:

- If $n = 0$, the set $\{x \in \mathbb{N} : x < n\} = \emptyset$, hence there can not be an injection from this set to itself which is not a bijection.
- Suppose the statement is true for $n = k$, and there is an injection h from $\{x \in \mathbb{N} : x < k + 1\}$ to itself which is not a surjection. We will now define a map $h' : \{x \in \mathbb{N} : x < k\} \to \{x \in \mathbb{N} : x < k\}$ as

  $$h'(j) = \begin{cases} h(k) & h(j) = k \\ h(j) & h(j) < k \end{cases}$$

  If $h(j) = k$ for some $j < k$, then injectivity of h would implies that $h(k) < k$, hence h' is well defined. It is also evident from definition that h' is injective. When $k \in Range(h)$, $Range(h') = Range(h) \cap \{x \in \mathbb{N} : x < k\}$; when $k \notin Range(h)$, $Range(h') = Range(h) \backslash \{h(k)\}$, hence in both cases h' can not be surjective, which contradicts with the inductive hypothesis.

3 implies 2; Let $C = \{A \backslash F : |F| = |\{x \in \mathbb{N} : x < n\}| \text{ for some } n \in \mathbb{N}\}$. Then 3 implies that $\emptyset \notin C$. By AC, there is a function $c : C \to \bigcup C = A$ such that $c(x) \in x$. Pick $a \in A$, define $f : \mathbb{N} \to A$ inductively as $f(0) = a$, $f(n+1) = c(A \backslash \{f(k) : k \in \mathbb{N}, k < n+1\})$. This is an injection due to the construction of c.

2 implies 1: Let $f : \mathbb{N} \to A$ be an injection, then $g : A \to A \backslash \{f(0)\}$ defined as

$$g(a) = \begin{cases} a & a \notin Range(f) \\ f(n+1) & a = f(n) \end{cases}$$

is a bijection. □

Definition 1.5.10. If a set A satisfies any of the 3 conditions in Theorem 1.5.9, we call it an **infinite set**. Otherwise we call it a **finite set**.

Definition 1.5.11. As shown in Theorem 1.5.9, a finite set A must have a bijection to a set of the form $\{x \in \mathbb{N} : x < n\}$. We can write $|A| = n$.

1.5.2 Countably infinite set

Definition 1.5.12. If $|A| = |\mathbb{N}|$, we call A a **countably infinite set**. If $|A| > |\mathbb{N}|$, we call A an **uncountable set**.

Remark 1.5.13. The following sets are countably infinite:

- The set of integers.
- The set of prime numbers.
- $\mathbb{N} \times \mathbb{N}$. A bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ can be defined as

$$(i, j) \mapsto (i + j)(i + j + 1)/2 + i$$

- The union of countably many countably infinite sets.
- The Cartesian product of finitely many countably infinite sets.
- The set of finite subsets of a countably infinite set.
- $\mathbb{Q}$.
- Set of polynomials with integer coefficients.
- Set of polynomials with rational coefficients.

2 Real Numbers

2.1 Ordered fields

Remark 2.1.1. Consider the set of rational numbers $\mathbb{Q}$, with two functions $+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ and $\times : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, and a linear order $\leq$. These three relations satisfy the following properties:

- $a + b = b + a$, $ab = ba$.
- $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$.
- $a + 0 = a$, $a \times 1 = a$.
- For any $a \in \mathbb{Q}$, there is some $-a \in \mathbb{Q}$ such that $(-a) + a = 0$.
- If $a \neq 0$, there is some $a^{-1} \in \mathbb{Q}$, such that $aa^{-1} = 1$.
- $a(b + c) = ab + ac$.
- $a \leq b$ then $a + c \leq b + c$.
- If $0 \leq a$, $0 \leq b$, then $0 \leq ab$.

Definition 2.1.2. A set F, with two distinct elements $0, 1 \in F$, two functions $+, \times : F \times F \to F$, and a linear order $\leq$, is called an **ordered field**, if it satisfies the conditions in the remark above. In other words,

- For any $a, b \in F$, $a + b = b + a$, $a \times b = b \times a$.
- For any $a, b, c \in F$, $(a + b) + c = a + (b + c)$, $(a \times b) \times c = a \times (b \times c)$, $a \times (b + c) = a \times b + a \times c$.
- For any $a \in F$, $a + 0 = a \times 1 = a$.
- For any $a \in F$, there is some $-a \in F$ such that $a + (-a) = 0$. If $a \neq 0$, there is some $a^{-1} \in F$ such that $a \times a^{-1} = 1$.
- If $a \leq b$ then $a + c \leq b + c$.
- If $0 \leq a$, $0 \leq b$ then $0 \leq a \times b$.

Remark 2.1.3. If $a \leq b$ and $a \neq b$ we write $a < b$.

Lemma 2.1.4. Let F be an ordered field, $a, b \in F$, then

- $a \leq b \implies -a \geq -b$.
- $a > b > 0 \implies b^{-1} > a^{-1} > 0$

Proof.

- Add $-a - b$ to both sides of the inequality $a \leq b$.
- Firstly we show that $a > 0$ implies $a^{-1} > 0$.

– Suppose $a > 0$ but $a^{-1} = 0$, then $0 = 0a = a^{-1}a = 1$, a contradiction.

– Suppose $a > 0$ but $a^{-1} < 0$, then $-1 = -a^{-1} \times a > 0$, hence $1 = (-1) \times (-1) > 0$ but also $0 = (-1) + 1 > 0 + 1 = 1$, also a contradiction.

Now we only need to show that $a > b > 0$ implies $b^{-1} > a^{-1}$. This is because $b^{-1} - a^{-1} = (a - b)(ab)^{-1} > 0$.

□

Definition 2.1.5. Let F be an ordered field, $a, b \in F$, $a < b$. We can define the concept of **intervals**:

$$(a, b) = \{r \in F : a < r, r < b\}$$

$$(a, b] = \{r \in F : a < r, r \leq b\}$$

$$[a, b) = \{r \in F : a \leq r, r < b\}$$

$$[a, b] = \{r \in F : a \leq r, r \leq b\}$$

The **length** of these intervals can be defined as $b - a$.

2.2 Least Upper Bound Property

Definition 2.2.1. We say that an ordered field F has the **least upper bound property**, if for any non-empty subset $A \subseteq F$, if there is some $m \in F$ such that for all $a \in A$, $a \leq m$, then there must be an element $\sup(A) \in F$, called the **least upper bound** or the **supremum** of A, that satisfies the following:

- For any $a \in A$, $a \leq \sup(A)$.
- If $m \in F$ satisfies that for any $a \in A$, $a \leq m$, then $\sup(A) \leq m$.

Theorem 2.2.2. There is an ordered field with least upper bound property that contains $\mathbb{Q}$ as a subfield (i.e. the 0 and 1 in this field are the 0 and 1 in $\mathbb{Q}$, $+, \times, \leq$ when restricted to $\mathbb{Q}$ are the same as those on $\mathbb{Q}$. We call it **the field of real numbers**, denoted as $\mathbb{R}$.

The proof can be found in most analysis textbooks.

Remark 2.2.3. The field of real numbers is **unique up to isomorphism**. In other words, given any two fields satisfying the Theorem above, there must be a bijection between them that sends 0 to 0, 1 to 1, and preserves $+, \times, \leq$.

Remark 2.2.4.

- Smallest upper bound property is equivalent to the "largest lower bound property", i.e. the fact that any non-empty subset with a lower bound has a largest lower bound, denoted as $\inf(A)$.
- If the smallest upper bound of a non-empty set A exist, it must be unique. Similarly for the largest lower bound.

2.3 Archimedean property

Theorem 2.3.1. (Archimedean property) Let $a, b \in \mathbb{R}$, $a > 0$, $b > 0$, then there must be some $n \in \mathbb{N}$ such that $na > b$.

Proof. Suppose this is not true, then there are $a, b > 0$ such that $na \leq b$ for all $n \in \mathbb{N}$. Now consider the non-empty set $S = \{na : a \in \mathbb{N}\}$, then b is an upper bound of S, hence S must have a least upper bound $m = \sup(S)$. The fact that m is an upper bound implies that for any $n \in \mathbb{N}$, $na \leq m$, hence $(n+1)a \leq m$, which implies $na \leq m - a$. Hence $m - a$ is another upper bound of S and $m - a < m$, this is a contradiction. □

Remark 2.3.2. Archimedean property is weaker than least upper bound property. For example $\mathbb{Q}$ has the former but not the latter.

As a consequence, we have

Theorem 2.3.3. Any real number can be uniquely written as the sum of an integer and a real number in $[0, 1)$.

Proof. Firstly we show the uniqueness of this decomposition. Suppose $x = n + r = n' + r'$, $n, n' \in \mathbb{Z}$, $r, r' \in [0, 1)$, then $n - n' = r' - r$. However $r' - r \in (-1, 1)$, $n - n' \in \mathbb{Z}$, hence $n - n' = 0$, $n = n'$ and $r = r'$.

Now we show the existence of such a decomposition.

- **Case 1**: If $x \in \mathbb{Z}$, then $x = x + 0$.
- **Case 2**: If $x > 0$ and $x \notin \mathbb{Z}$, Theorem 2.3.1 implies that there must be natural number larger than x. Let n be the **smallest such natural number**, then $n - 1 < x < n$, $x = (n-1) + (x - n + 1)$, $n - 1 \in \mathbb{Z}$, $x - n + 1 \in (0, 1)$.
- **Case 3**: If $x < 0$ and $x \notin \mathbb{Z}$, then $-x$ satisfies Case 2 above, hence $-x = n + r$ where $n \in \mathbb{Z}$ and $r \in (0, 1)$. So $x = (-n-1) + (1-r)$, $-n - 1 \in \mathbb{Z}$ and $1 - r \in [0, 1)$.

□

Remark 2.3.4. In the proof above we used the key property of $\mathbb{N}$ that any non-empty subset has a smallest element. This is equivalent to mathematical induction.

Remark 2.3.5. If $x = n + r$, $n \in \mathbb{Z}$ and $r \in [0, 1)$, we denote n as $\lfloor x \rfloor$.

Theorem 2.3.6. ("$\mathbb{Q}$ is dense in $\mathbb{R}$") For any $x \in \mathbb{R}$, any $\epsilon > 0$, there is some $q \in \mathbb{Q} \cap (x - \epsilon, x + \epsilon)$.

Proof.
- If $x = 0$, then let $q = 0$.

- If $x > 0$, by Theorem 2.3.1 there is a positive natural number n such that $n > 1/\epsilon > 0$. Hence $0 < 1/n < \epsilon$. Let m be the smallest natural number such that $m/n > x$ (m exists due to Theorem 2.3.1), then $(m-1)/n \leq x < x + \epsilon$, and $(m-1)/n = m/n - 1/n > x - 1/n > x - \epsilon$. Hence $q = (m-1)/n$ is in $(x - \epsilon, x + \epsilon) \cap \mathbb{Q}$.

- If $x < 0$, apply the Case above to $-x$, we can find some $q \in (-x - \epsilon, -x + \epsilon) \cap \mathbb{Q}$, hence $-q \in (x - \epsilon, x + \epsilon) \cap \mathbb{Q}$.

□

Remark 2.3.7. If $a, b \in \mathbb{R}$, $a < b$, then apply the theorem above for $x = (a+b)/2$ and $\epsilon = (b-a)/2$, we know that $(a, b) \cap \mathbb{Q} \neq \emptyset$.

Remark 2.3.8. As a consequence, the map from $\mathbb{R}$ to $P(\mathbb{Q})$ defined as $r \mapsto \{q \in \mathbb{Q} : q \leq r\}$ is an injection. Hence $|\mathbb{R}| \leq |P(\mathbb{Q})|$.

2.4 Infinite Decimal Expansions

We will show that any real number in $[0, 1)$ has an **infinite decimal expansion**. More precisely:

Theorem 2.4.1. There is a bijection $\mathcal{F}$, from real numbers in $[0, 1)$, to the set

$$S = \{a \in \{0, 1, \ldots, 9\}^{\mathbb{N}} : \text{ for any } n, \text{ there exists } m > n, a(m) < 9\}$$

defined as

$$x \mapsto (n \mapsto \lfloor 10(10^n x - \lfloor 10^n x \rfloor) \rfloor)$$

Proof. **Step 1**: Show that $\mathcal{F}$ is well defined. In other words, show that

$$(n \mapsto \lfloor 10(10^n x - \lfloor 10^n x \rfloor) \rfloor) \in S$$

By construction,

$$10^n - \lfloor 10^n \rfloor \in [0, 1)$$

$$10(10^n x - \lfloor 10^n x \rfloor) \in [0, 10)$$

Hence

$$\lfloor 10(10^n x - \lfloor 10^n x \rfloor) \rfloor \in \{0, \ldots, 9\}$$

Next we need to show that the function $(n \mapsto \lfloor 10(10^n x - \lfloor 10^n x \rfloor) \rfloor)$ must take values less than 9 infinite many times. To do so, we need the following lemma:

Lemma 2.4.2. Let $y_n = 10^n x - \lfloor 10^n x \rfloor$. Then for any $n \in \mathbb{N}$,

$$y_{n+1} = 10 y_n - \lfloor 10 y_n \rfloor$$

Proof. The definition of $\lfloor\cdot\rfloor$ implies that for any $r \in \mathbb{R}$, any $n \in \mathbb{Z}$,

$$\lfloor r+n \rfloor = \lfloor r \rfloor + n$$

Hence

$$y_{n+1} = 10(10^n x) - \lfloor 10(10^n x) \rfloor$$
$$= 10(\lfloor 10^n x \rfloor + y_n) - \lfloor 10(\lfloor 10^n x \rfloor + y_n) \rfloor = 10y_n - \lfloor 10y_n \rfloor$$

□

Now, suppose there is some $x \in [0,1)$, $\mathcal{F}(x) \notin S$. In other words, there is some $n_0 \in \mathbb{N}$, such that for any $m > n_0$, $(\mathcal{F}(x))(m) = 9$. Lemma 2.4.2 implies that if we define $y_n = 10^n x - \lfloor 10^n x \rfloor$, then

$$y_{m+1} = 10y_m - \lfloor 10y_m \rfloor = 10y_m - (\mathcal{F}(x))(m)$$

Hence, if $(\mathcal{F}(x))(m) = 9$, then

$$1 - y_{m+1} = 10(1 - y_m)$$

Hence we have $1 - y_{n_0+1} \in (0,1]$ and

$$10^k(1 - y_{n_0+1}) = 1 - y_{n_0+1+k} \in (0,1]$$

for all $k \in \mathbb{N}$, which contradicts with Theorem 2.3.1.

Step 2: Injectivity of $\mathcal{F}$. Let $x, x' \in [0,1)$, $x < x'$. Suppose $\mathcal{F}(x) = \mathcal{F}(x')$, then if we define

$$y_n = 10^n x - \lfloor 10^n x \rfloor$$
$$y'_n = 10^n x' - \lfloor 10^n x' \rfloor$$

Then Lemma 2.4.2 implies that

$$y'_{n+1} - y_{n+1} = 10(y'_n - y_n) - (\lfloor 10y'_n \rfloor - \lfloor 10y_n \rfloor)$$
$$= 10(y'_n - y_n) - ((\mathcal{F}(x'))(n) - (\mathcal{F}(x))(n)) = 10(y'_n - y_n)$$

Hence $y'_n - y_n = 10^n(y'_0 - y_0) = 10^n(x' - x)$ for all $n \in \mathbb{N}$, while $x' - x > 0$ and $y'_n - y_n \le 1$. This contradicts with Theorem 2.3.1.

Step 3: Surjectivity of $\mathcal{F}$. Given $w \in S$, let x_w be the least upper bound of the set

$$A_w = \left\{ \sum_{n=0}^{m} \frac{w(n)}{10^{1+n}} : n \in \mathbb{N} \right\}$$

Given any $k \in \mathbb{N}$, by assumption on S, there must be some $k' > k$ such that $w(k') < 9$. The fact that x_w is an upper bound implies that

$$x_w \ge \sum_{n=0}^{k} \frac{w(n)}{10^{1+n}}$$

while the fact that x_w is the smallest upper bound means that it can be no larger than another upper bound of A_w which is $\sum_{n=0}^{k} \frac{w(n)}{10^{1+n}} + \frac{1}{10^{k+1}} - \frac{1}{10^{1+k'}}$, hence

$$x_w < \sum_{n=0}^{k} \frac{w(n)}{10^{1+n}} + \frac{1}{10^{k+1}}$$

These two inequalities imply that $(\mathcal{F}(x_w))(k) = w(k)$. □

Remark 2.4.3. $|S| \leq |P(\mathbb{N})|$ because S is a subset of the power set $\mathbb{N} \times \{0, \ldots, 9\}$ which has the same cardinality as $\mathbb{N}$. There is also an injection $f : P(\mathbb{N}) \to S$ defined as

$$(f(A))(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases}$$

Hence the cardinality of $\mathbb{R}$, which is the same as the cardinality of $[0, 1)$, is $|P(\mathbb{N})|$.

Remark 2.4.4. The Remark above implies that there must be irrational numbers. Actually, Fermat's Little Theorem implies that the infinite decimal expansion of any rational number must be eventually periodic.

2.5 Other Consequences of the Least Upper Bound Property

Theorem 2.5.1. Let $I_k = [a_k, b_k]$, $k \in \mathbb{N}$, $b_k \geq a_k$, be a sequence of finite closed intervals in $\mathbb{R}$ that satisfies $I_{k+1} \subseteq I_k$. Then $\bigcap_{k \in \mathbb{N}} I_k \neq \emptyset$. Furthermore, if for any $\epsilon > 0$, there is some k such that $b_k - a_k < \epsilon$, then $\bigcap_{k \in \mathbb{N}} I_k$ consists of a unique element.

Proof. Let $A = \{a_k : k \in \mathbb{N}\}$. For any $j \in \mathbb{N}$, if a natural number k is no more than j then $a_k \leq a_j \leq b_j$, and if $k > j$ then $a_k \leq b_k \leq b_j$. Hence b_j is an upper bound of A. The smallest upper bound property implies that there must be some real number $m = \sup(A)$. m is an upper bound of A implies $m \geq a_j$ for all $j \in \mathbb{N}$, and the minimality of m among the upper bounds of A implies $m \leq b_j$ for all $j \in \mathbb{N}$, hence $m \in I_j$ for all $j \in \mathbb{N}$.

Furthermore, suppose $x, y \in \bigcap_{k \in \mathbb{N}} I_k$, then $b_j - a_k \geq y - x$ for all $k \in \mathbb{N}$. This proves the second part of the theorem. □

Theorem 2.5.2. Let $a < b$ be two real numbers, $I = [a, b]$. Let C be a set of open intervals such that $I \subseteq \bigcup C$. Then there must be a finite subset $C' \subseteq C$ such that $I \subseteq \bigcup C'$.

Proof. Firstly, by intersecting with $(a - 1, b + 1)$, we can assume all intervals in C are of finite length without loss of generality. In other words, any element in C is of the form (c, d) where $d > c$.

Assume the Theorem is not true, then there is an infinite set of finite open intervals C, such that $I \subseteq \bigcup C$ but any finite subset C' of C, $I \nsubseteq C'$. We call an interval J to have property $(*)$ if $J \subseteq \bigcup C$ and $J \nsubseteq \bigcup C'$ for any finite subset C' of C. Now we define a sequence of closed intervals inductively as below:

- $I_0 = I$.
- Let $I_n = [a_n, b_n]$. If $[a_n, (a_n + b_n)/2]$ satisfies property $(*)$, then $I_{n+1} = [a_n, (a_n + b_n)/2]$. Otherwise $I_{n+1} = [(a_n + b_n)/2, b_n]$

Now we use induction to show that all I_n satisfy property $(*)$. $I_0 = I$ satisfies $(*)$ is by assumption. If $I_k = [a_k, b_k]$ satisfies property $(*)$ and I_{k+1} doesn't, by construction the only possibility can only be that $[a_k, (a_k + b_k)/2]$ doesn't satisfy property $*$ and $I_{k+1} = [(a_k + b_k)/2, b_k]$. Let C' be a finite subset of C such that $[a_k, (a_k + b_k)/2] \subseteq \bigcup C'$, C'' a finite subset of C such that $I_{k+1} = [(a_k + b_k)/2, b_k] \subseteq \bigcup C''$, then

$$I_k = [a_k, (a_k + b_k)/2] \cup I_{k+1} = \bigcup (C' \cup C'')$$

and $C' \cup C''$ is a finite subset of C, which contradicts with the inductive hypothesis that I_k satisfies property $(*)$.

Now apply Theorem 2.5.1 to $\{I_n\}$, we get some $x \in \bigcap_{n \in \mathbb{N}} I_n$. Let $J = (c, d) \in C$ be an open interval that contains x, let $r = \min(d - x, x - c)$, then $r > 0$, hence there must be some $N \in \mathbb{N}$ such that $2^N \epsilon > b - a$. Because I_N is an interval that contains x and has length $2^{-N}(b - a) < r$, $I_n \subseteq (x - r, x + r) \subseteq J$, in other words $I_N \subseteq \bigcup\{J\}$, which contradicts with the fact that I_N satisfies property $(*)$. □

Theorem 2.5.3. Let $I = [a, b]$ be a finite closed interval, $Y \subseteq I$ an infinite subset, then there must be some $x \in I$ such that for any $\epsilon > 0$, $Y \cap (x - \epsilon, x + \epsilon)$ is an infinite set.

Proof. Suppose this is not true, then there is some closed interval $I = [a, b]$, some infinite set $Y \subseteq I$, such that for any $x \in I$, there is some r_x such that $Y \cap (x - r_x, x + r_x)$ is a finite set. Because $I \subseteq \bigcup_{x \in I}(x - r_x, x + r_x)$, apply Theorem 2.5.2 we get that there must be finitely many $x_1, \ldots, x_n \in I$ such that $[a, b] = \bigcup_{i=1}^{n}(x_i - r_{x_i}, x_i + r_{x_i})$. Hence

$$Y = \bigcup_{i=1}^{n}(Y \cap (x_i - r_{x_i}, x_i + r_{x_i}))$$

is the union of finitely many finite set, hence must be a finite set, which is a contradiction. □

2.6 Extended Real Numbers

Definition 2.6.1. The **extended real number system** is the set $\mathbb{R} \cup \{\pm\infty\}$, with a linear order where ∞ is defined as the largest and $-\infty$ as the smallest. We sometimes also use the following conventions for arithmetic involving $\pm\infty$. Here $a \in \mathbb{R}$.

- $a + (\pm\infty) = \pm\infty$.
- $\frac{a}{\pm\infty} = 0$.
- If $a > 0$, $a \times (\pm\infty) = \pm\infty$. If $a < 0$, $a \times (\pm\infty) = \mp\infty$.

Remark 2.6.2. The linear order on $\mathbb{R} \cup \{\pm\infty\}$ enables us to write **infinite intervals**, for example $(-\infty, a] = \{r \in \mathbb{R} : r \leq a\}$.

Remark 2.6.3. The least upper bound property on $\mathbb{R}$ implies the least upper bound property on $\mathbb{R} \cup \{\pm\infty\}$. So we can extend the definition of sup and inf as follows:

- If a subset A doesn't have an upper bound, we can write $\sup(A) = \infty$.
- If a subset A doesn't have a lower bound, we can write $\inf(A) = -\infty$.

Remark 2.6.4. We can also consider sets like $\mathbb{Z} \cup \{\pm\infty\}$ or $\mathbb{N} \cup \{\infty\}$.

3 Metric Spaces

3.1 Definition and Examples

Definition 3.1.1. A **metric space** is a pair (X, d), where X is a set and d is a function $d : X \times X \to [0, \infty)$, such that for any $p, q, r \in X$,

- $d(p, q) = 0$ iff $p = q$.
- (Symmetry) $d(p, q) = d(q, p)$.
- (Triangle Inequality) $d(p, q) + d(q, r) \geq d(p, r)$.

This function d is called the **metric** or **distance** function.

Remark 3.1.2. When (X, d) is a metric space, we can also write "X is a metric space with metric (or distance function) d".

Definition 3.1.3. Let (X, d) be a metric space.

- The **diameter** of X is defined as

  $$diam(X) = \sup(\{d(x, y) : x, y \in X\})$$

  If the range of d doesn't have upper bound we say the diameter is ∞. If $diam(X)$ is finite we also call it **bounded**.

- A **closed ball** centered at $x \in X$ with **radius** $r > 0$ is the set

  $$\{y \in X : d(x, y) \leq r\}$$

- An **open ball** centered at $x \in X$, with **radius** $r > 0$, denoted as $B_X(x, r)$ (or $B_{(X,d)}(x, r)$ when we want to specify the metric), is the set

  $$B_X(x, r) = \{y \in X : d(x, y) < r\}$$

Example 3.1.4. 1. Let X be a non-empty set,

$$d(p, q) = \begin{cases} 0 & p = q \\ 1 & p \neq q \end{cases}$$

Then (X, d) is a metric space. This d is called the **discrete metric**.

2. $\mathbb{R}$ with **Euclidean metric** $d(x, y) = |x - y|$ is a metric space.

3. $\mathbb{R}^n$ with **Euclidean metric**

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

is a metric space.

4. Let S be a finite set, $(P(S), (A, B) \mapsto |A \backslash B| + |B \backslash A|)$ is a metric space.

Remark 3.1.5. When we discuss $\mathbb{R}$ or $\mathbb{R}^n$ as metric spaces and do not explicitly specify the metric, we always assume the Euclidean metric.

Remark 3.1.6. If (X, d) is a metric space, $A \subseteq X$, then $(A, d|_{A \times A})$ is also a metric space, which we call a **subspace**.

Remark 3.1.7. It is evident, by triangle inequality, that an open ball or a closed ball with radius r has diameter no more than $2r$.

Theorem 3.1.8. If (X, d) is a metric space, $g : [0, \infty) \to [0, \infty)$ satisfies

- $g(t) = 0 \iff t = 0$
- $t \leq t' \implies g(t) \leq g(t')$
- $g(t + t') \leq g(t) + g(t')$

Then $(X, g \circ d)$ is also a metric space.

Proof.
- $g(d(x, y)) = 0 \iff d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$, hence $g(d(x, y)) = g(d(y, x))$.
- $d(x, z) \leq d(x, y)+d(y, z)$, hence $g(d(x, z)) \leq g(d(x, y)+d(y, z)) \leq g(d(x, y))+ g(d(y, z))$.

□

Example 3.1.9. $(\mathbb{R}, (x, y) \mapsto \min(1, |x - y|))$ and $(\mathbb{R}, (x, y) \mapsto \frac{|x-y|}{1+|x-y|})$ are both metric spaces with diameter 1.

Theorem 3.1.10. If (X, d) and (Y, d) are two metric spaces, let $d_{sup} : (X \times Y) \times (X \times Y) \to [0, \infty)$ be defined as

$$d_{sup}((x, y), (x', y')) = \max(d(x, x'), d'(y, y'))$$

Then $(X \times Y, d_{sup})$ is a metric space. d_{sup} is called the **sup metric**.

Proof.
- For any $x, x' \in X$, $y, y' \in Y$,

$$d_{sup}((x, y), (x', y')) = 0 \iff \max(d(x, x'), d'(y, y')) = 0$$

$$\iff d(x, x') = d'(y, y') = 0 \iff x = x' \text{ and } y = y' \iff (x, y) = (x', y')$$

- For any $x, x' \in X$, $y, y' \in Y$,

$$d_{sup}((x, y), (x', y')) = \max(d(x, x'), d'(y, y'))$$

$$= \max(d(x', x), d'(y', y)) = d_{sup}((x', y'), (x, y))$$

- For any $x_1, x_2, x_3 \in X$, $y_1, y_2, y_3 \in Y$, for $i = 1, 2$,

  $$d(x_i, x_{i+1}) \leq \max(d(x_i, x_{i+1}), d'(y_i, y_{i+1})) \leq d_{sup}((x_i, y_i), (x_{i+1}, y_{i+1}))$$

  $$d'(y_i, y_{i+1}) \leq \max(d(x_i, x_{i+1}), d'(y_i, y_{i+1})) \leq d_{sup}((x_i, y_i), (x_{i+1}, y_{i+1}))$$

  So

  $$d(x_1, x_2) + d(x_2, x_3) \leq d_{sup}((x_1, y_1), (x_2, y_2)) + d_{sup}((x_2, y_2), (x_3, y_3))$$

  $$d'(y_1, y_2) + d'(y_2, y_3) \leq d_{sup}((x_1, y_1), (x_2, y_2)) + d_{sup}((x_2, y_2), (x_3, y_3))$$

  Hence

  $$\begin{aligned} d_{sup}((x_1, y_1), (x_3, y_3)) &= \max(d(x_1, x_3), d'(y_1, y_3)) \\ &\leq \max(d(x_1, x_2) + d(x_2, x_3), d'(y_1, y_2) + d'(y_2, y_3)) \\ &\leq d_{sup}((x_1, y_1), (x_2, y_2)) + d_{sup}((x_2, y_2), (x_3, y_3)) \end{aligned}$$

□

Example 3.1.11. Repeatedly apply Theorem 3.1.10 to $(\mathbb{R}, (x, y) \mapsto |x - y|)$, we have $(\mathbb{R}^n, d_{sup})$ where

$$d_{sup}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\{|x_i - y_i| : i = 1, \ldots, n\}$$

is a metric space.

Theorem 3.1.12. Let S be a non-empty set, (X, d) a metric space. Define

$$BF(S, X) = \{f \in Map(S, X) : diam(f(S)) < \infty\}$$

Then $BF(S, X)$ is a metric space under the **sup metric**

$$d_{sup}(f, g) = \sup(\{d(f(s), g(s)) : s \in S\})$$

Proof.

- d_{sup} is well defined, because the non-emptyness of S implies that $\{d(f(s), g(s)) : s \in S\}$ is non-empty, and if we pick $s_0 \in S$, then this set is bounded from above by

  $$diam(f(S)) + d(f(s_0), g(s_0)) + diam(g(S)) < \infty$$

  Hence d_{sup} exists due to smallest upper bound property.

- $d_{sup}(f, g) = 0$ iff for all $s \in S$, $d(f(s), g(s)) = 0$ iff $f = g$.

- $d_{sup}(f, g) = \sup(\{d(f(s), g(s)) : s \in S\}) = \sup(\{d(g(s), f(s)) : s \in S\}) = d_{sup}(g, f)$.

- For any $f_1, f_2, f_3 \in BF(S, X)$, for any $s \in S$, $i = 1, 2$, the definition of d_{sup} implies that

$$d(f_i(s), f_{i+1}(s)) \leq d_{sup}(f_i, f_{i+1})$$

Hence for any $s \in S$,

$$d(f_1(s), f_3(s)) \leq d(f_1(s), f_2(s)) + d(f_2(s), f_3(s)) \leq d_{sup}(f_1, f_2) + d_{sup}(f_2, f_3)$$

In other words, $d_{sup}(f_1, f_2) + d_{sup}(f_2, f_3)$ is an upper bound of $\{d(f_1(s), f_3(s)) : s \in S\}$, hence $d_{sup}(f_1, f_2) + d_{sup}(f_2, f_3) \geq \sup(\{d(f_1(s), f_3(s)) : s \in S\}) = d_{sup}(f_1, f_3)$.

□

Remark 3.1.13. As one can see from the examples above, a set can have many different metrics. For example $\mathbb{R}^n$ can have the discrete metric, the sup metric (Example 3.1.11), the Euclidean metric. and many more.

3.2 Open Sets and Closed Sets

3.2.1 Definition and Examples

Definition 3.2.1. Let (X, d) be a metric space.

- A subset $A \subseteq X$ is called **open**, if for any $a \in A$, there is some $r_a > 0$ such that the open ball $B_X(a, r_a)$ is a subset of A.
- A subset $A \subseteq X$ is called **closed**, if $X \backslash A$ is open.

Remark 3.2.2. The definition of closed set above can also be rewritten as: $A \subseteq X$ is closed iff for any $x \in X \backslash A$, there is some $r_x > 0$ such that $B_X(x, r_x) \cap A = \emptyset$.

Remark 3.2.3. The "openness" or "closedness" of a set depends on the following data: the metric space X. the metric d on X, and the subset $A \subseteq X$. For example, $(0, 1]$ is not open in $\mathbb{R}$ with Euclidean metric but is open in $\mathbb{R}$ with discrete metric, and is also open in $(-\infty, 1]$ with Euclidean metric $(x, y) \mapsto |x - y|$.

Example 3.2.4. If (X, d) is a metric space, then X and $\emptyset$ are both open and closed.

Example 3.2.5. If (X, d_{disc}) is a metric space with discrete metric, then any subset of X is both open and closed.

Theorem 3.2.6. Any open ball is open.

Proof. Let (X, d) be a metric space, $x \in X$, $r > 0$. For any $p \in B_X(x, r)$, let $r' = r - d(x, p) > 0$, then for any $q \in B_X(p, r')$,

$$d(q, x) \leq d(q, p) + d(p, x) < r' + d(x, p) = r$$

Hence $B_X(p, r') \subseteq B_X(x, r)$. □

Theorem 3.2.7. Any closed ball is closed. The set consisting of a single point is closed.

Proof. Let x be a point in metric space (X, d), $r > 0$, let B be the closed ball centered at x with radius r. For any $y \in X \backslash B$, by definition of closed balls $d(x, y) > r$, hence triangle inequality implies $B_X(y, d(x, y) - r) \cap B = \emptyset$. Similarly, for any $x \in X$, any $y \in X \backslash \{x\}$, $B_X(y, d(x, y)) \cap \{x\} = \emptyset$. $\square$

3.2.2 Basic Properties

Theorem 3.2.8. The union of a set of open sets is open.

Proof. Let (X, d) be a metric space, C a set of open sets. For any $x \in \bigcup U$, there is some $U \in C$ such that $x \in U$ and U is also open, hence there is some $r > 0$ such that $B_X(x, r) \subset U \subseteq \bigcup C$. $\square$

Remark 3.2.9. An immediate consequence of Theorem 3.2.8 is that intersection of any non-empty set of closed sets is closed.

This enables us to define the following:

Definition 3.2.10. Let (X, d) be a metric space, $A \subseteq X$. We define

- The **closure** of A as $\overline{A} = \bigcap\{V \subseteq X : V \text{ closed}, A \subseteq V\}$.
- The **interior** of A as $A^\circ = \bigcup\{U \subseteq X : U \text{ open}, U \subseteq A\}$.
- The **boundary set** of A as $\partial A = \overline{A} \backslash A^\circ$.
- A point in the interior is called an **interior point**, a point in the boundary set is called a **boundary point**.

Remark 3.2.11.

- $\overline{A}$ is the smallest closed set that has A as a subset. A° is the largest open set that is a subset of A.
- $X \backslash A^\circ = \overline{X \backslash A}$, $\partial A = \overline{A} \cap \overline{X \backslash A}$.
- One can characterize these sets with open balls as follows:
  - $p \in \overline{A}$ iff for any $r > 0$, $B_X(p, r) \cap A \neq \emptyset$.
  - $p \in A^\circ$ iff there is some $r > 0$, $B_X(p, r) \subseteq A$.
  - $p \in \partial A$ iff for any $r > 0$, $B_X(p, r) \cap A \neq \emptyset$, $B_X(p, r) \cap (X \backslash A) \neq \emptyset$.

Theorem 3.2.12. The intersection of finitely many open sets is open.

Proof. Let U_i, $i = 1, 2, \ldots, n$ be finitely many open subsets of metric space (X, d). Let $U = \bigcap_{i=1}^n U_i$. For every $p \in U$, for every $i = 1, 2, \ldots, n$, there is $r_i > 0$ such that $B_X(p, r_i) \subseteq U_i$. Hence if we set $r = \min(r_i) > 0$ then $B_X(p, r) \subseteq U$. $\square$

Remark 3.2.13. As a consequence of Theorem 3.2.12, the union of finitely many closed sets must be closed. In particular, any finite subset of a metric space is closed.

Remark 3.2.14. The intersection of infinitely many open sets may not necessarily be open. For example, $\{x\} = \bigcap_{n=1}^{\infty} B_X(p, 1/n)$ but in general $\{x\}$ is not always open.

Example 3.2.15. The closure of an open ball of radius r is not necessarily the closed ball of radius r. For example, consider $X = (-\infty, 0] \cup [1, \infty)$ with Euclidean metric, $B_X(1, 1) = [1, 2)$, its closure is $[1, 2]$, but the closed ball centered at 1 with radius 1 is $\{0\} \cup [1, 2]$.

Theorem 3.2.16. A subset of a metric space (X, d) is open iff it is a union of open balls.

Proof. Theorem 3.2.6 and Theorem 3.2.8 implies that union of open balls is open. If $A \subseteq X$ is open, for every $x \in A$, there is some $r_x > 0$ such that $B_X(x, r_x) \subseteq A$. Hence $A = \bigcup_{x \in A} B(x, r_x)$. □

3.2.3 Openness in subspaces

An application of Theorem 3.2.16 is the following:

Theorem 3.2.17. If (X, d) is a metric space, $(Y, d|_{Y \times Y})$ a subspace, $A \subseteq Y$ is open in Y iff there is some open set A' in X such that $A = A' \cap Y$. Similarly, $A \subseteq Y$ is closed in Y iff there is some closed set A' in X such that $A = A' \cap Y$.

Proof. For any $y \in Y$, any $r > 0$

$$B_Y(y, r) = \{p \in Y : d(y, p) < r\} = Y \cap \{p \in X : d(y, p) < r\} = Y \cap B_X(y, r)$$

If A is open in Y, for every $a \in A$ there is some $r_a > 0$ such that $B_Y(a, r_a) \subseteq A$. Let $A' = \bigcup_{a \in A} B_X(a, r_a)$, then A' is open in X by Theorem 3.2.16 and $A' \cap Y = \bigcup_{a \in A}(B_X(a, r_a) \cap Y) = \bigcup_{a \in A} B_Y(a, r_a) = A$. The statement about closed sets follows from the one about open sets. □

3.2.4 Denseness

Definition 3.2.18. A subset $A \subseteq X$ is called **dense** if any non-empty open subset of X has non-empty intersection with A.

Remark 3.2.19. If $A \subseteq B \subseteq X$ and A is a dense subset of B under subspace metric, we can also say "A is dense in B".

Example 3.2.20. Theorem 2.3.6 shows that $\mathbb{Q}$ is dense in $\mathbb{R}$ under Euclidean metric.

Remark 3.2.21. Remark 3.2.11 implies that A is always dense in $\overline{A}$. Actually, we have $\overline{A} = \bigcup\{B \subseteq X : A \subseteq B, A \text{ is dense in } B\}$.

3.2.5 Open sets and closed sets in $\mathbb{R}$

In this section we will only focus on metric space $\mathbb{R}$ under Euclidean metric $d(x, y) = |x - y|$. It is easy to see that $B_{\mathbb{R}}(x, r) = (x - r, x + r)$. Hence we have:

Lemma 3.2.22. Any finite open interval of the form (a, b), where $b > a$, or any infinite open interval of the form $(-\infty, a)$, (a, ∞) or $(-\infty, \infty)$, must be open.

Proof. By Theorem 3.2.16 we only need to write them into unions of open balls:

- $(a, b) = B_{\mathbb{R}}((a + b)/2, (b - a)/2)$.
- $(-\infty, a) = \bigcup_{n \in \mathbb{N}} B_{\mathbb{R}}(a - n - 1, n + 1)$.
- $(a, \infty) = \bigcup_{n \in \mathbb{N}} B_{\mathbb{R}}(a + n + 1, n + 1)$.
- $(-\infty, \infty) = \bigcup_{n \in \mathbb{N}} B_{\mathbb{R}}(0, n + 1)$.

□

Remark 3.2.23. A consequence of Lemma 3.2.22 is that any (finite or infinite) closed interval must be closed.

Theorem 3.2.24. If $A \subseteq \mathbb{R}$ is non-empty and $\sup(A) < \infty$, then

- If A is open then $\sup(A) \notin A$.
- If A is closed then $\sup(A) \in A$.

Proof.

- Suppose A is open and $\sup(A) \in A$, then there is some $r > 0$ such that $(\sup(A) - r, \sup(A) + r) \subseteq A$, hence $\sup(A) + r/2 \in A$ and $\sup(A) + r/2 > \sup(A)$, a contradiction.
- Suppose A is closed and $\sup(A) \notin A$, then $A \subseteq (-\infty, \sup(A))$. The closedness of A implies that there is some $r > 0$ such that $(\sup(A) - r, \sup(A) + r) \cap A = \emptyset$, hence $(\sup(A) - r, \infty) \cap A = ((\sup(A) - r, \sup(A) + r) \cap [\sup(A), \infty)) \cap A = \emptyset$, $\sup(A) - r$ is an upper bound of A which is smaller than $\sup(A)$, a contradiction.

□

Remark 3.2.25. Replacing sup with inf in Theorem 3.2.24 is still true, the proof is almost identical.

Theorem 3.2.26. A subset A of $\mathbb{R}$ is open under Euclidean metric iff it is the disjoint union of finitely many or countably infinitely many (finite or infinite) open intervals. More precisely, if A is open then $A = \bigcup C$, where elements in C are open intervals, $|C| \leq |\mathbb{N}|$, and for any two $I, I' \in C$, if $I \neq I'$ then $I \cap I' = \emptyset$.

Proof. The "if" part follows from Lemma 3.2.22 and Theorem 3.2.8. We will now focus on the "only if" part.

Step 1: Build the set C. For every $a \in A$, consider $V = [a, \infty)\backslash A$ and $V' = (-\infty, a]\backslash A$. If $V = \emptyset$ set $M_a = \infty$, otherwise $M_a = \inf(V)$. If $V' = \emptyset$ set $m_a = -\infty$, otherwise $m_a = \sup(V')$. And let $C = \{(m_a, M_a) : a \in A\}$.

Step 2: Show that $A \subseteq \bigcup C$. Because A is open, for every $a \in A$, there is some $r > 0$ such that $(a - r, a + r) \subseteq A$. Hence $V = [a, \infty)\backslash A \subseteq [a, \infty)\backslash(a - r, a + r) = [a + r, \infty)$, which implies that $M_a \geq a + r > a$, Similarly we can show that $m_a < a$, hence $a \in (m_a, M_a)$, which implies $A \subseteq \bigcup C$.

Step 3: Show that $\bigcup C \subseteq A$. To show this we only need to show that for every $a \in A$, $(m_a, M_a) \subseteq A$. Let y be any element in $\mathbb{R}\backslash A$, because $a \in A$, $y < a$ or $y > a$. If $y > a$ then $y \in V$, $V \neq \emptyset$, hence $M_a = \inf(V) \leq y$, $y \notin (m_a, M_a)$. The case when $y < a$ is analogous. Hence $(m_a, M_a) \cap (\mathbb{R}\backslash A) = \emptyset$, in other words $(m_a, M_a) \subseteq A$.

Step 4: Show that $m_a, M_a \notin A$. This follows from Theorem 3.2.24.

Step 5: Show that if $I, I' \in C$, $I \neq I'$, then $I \cap I' = \emptyset$. Suppose $(m_a, M_a) \cap (m_b, M_b) \neq \emptyset$, let y be an element in their intersection. Suppose $m_a < m_b$, then $m_b \in (m_a, y) \subseteq (m_a, M_a) \subseteq A$, the last $\subseteq$ due to Step 3. However $m_b \notin A$ due to Step 4, which is a contradiction. Similarly $m_b < m_a$, $M_a < M_b$ or $M_b < M_a$ are all impossible, hence $(m_a, M_a) = (m_b, M_b)$.

Step 6: Show that $|C| \leq |\mathbb{N}|$. For every $I \in C$ we can pick some $q_i \in I \cap \mathbb{Q}$, then $I \mapsto q_I$ is an injection from C to $\mathbb{Q}$. □

Remark 3.2.27. Open subsets of $\mathbb{R}$ can be complicated, for example, let i be any bijection from $\mathbb{N}$ to $\mathbb{Q}$, then $U = \bigcup_{n \in \mathbb{N}} (i(n) - 2^{-n}, i(n) + 2^{-n})$ is a dense open set in $\mathbb{R}$ but the total length of the disjoint open intervals that form U is no more than 4.

3.3 Continuity

3.3.1 Definitions and Basic Examples

Definition 3.3.1. Let (X, d) and (Y, d') be two metric spaces, $f : X \to Y$ a map.

- We call f to be **continuous** if for any open set $U \subseteq Y$, its preimage $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is open in X.
- We call f to be **continuous at** $x \in X$ if for any open set $U \subseteq Y$, if $f(x) \in U$, then there is an open set $V \subseteq X$ such that $x \in V$ and $f(V) \subseteq U$.

Remark 3.3.2. The definition of continuity above is equivalent to "preimages of closed sets are closed".

Example 3.3.3. • Constant maps are continuous.

- Any map from a metric space with discrete metric to another metric space is continuous.
- Identity map is continuous as a map from (X, d) to (X, d). However, if $d \neq d'$, id_X is not necessarily continuous as a map from (X, d) to (X, d'). For example $X = \mathbb{R}$, d is the Euclidean metric and d' the discrete metric.
- If (X, d) is a metric space, $A \subseteq X$, the inclusion map is continuous as a map from $(A, d|_{A\times A})$ to (X, d).
- If (X, d), (Y, d') are metric spaces, the map $(x, y) \mapsto x$ is continuous from $(X \times Y, d_{sup})$ to (X, d). Here d_{sup} is as defined in Theorem 3.1.10.
- If (X, d) is a metric space, S a non-empty set, $s \in S$, then $f_s : f \mapsto f(s)$ is continuous from $(BF(S, X), d_{sup})$ as defined in Theorem 3.1.12 to (X, d).

Remark 3.3.4. From definition, it is evident that compositions of continuous maps must be continuous. And if $f : X \to Y$ is continuous at $x \in X$, $g : Y \to Z$ continuous at $f(x)$, then $g \circ f$ is continuous at x. As a consequence, we have

- If $f : X \to Y$ is continuous, $Z \subseteq X$ a subspace, then $f|_Z : Z \to Y$ is continuous as well.
- Let $f : X \to Y$, $f(X) \subseteq Z \subseteq Y$, then f seen as a map from X to Y is continuous iff it is continuous as a map from X to Z. Here the "only if" part follows from Theorem 3.2.17.

3.3.2 Localness and $\epsilon - \delta$

Theorem 3.3.5. $f : X \to Y$ is a continuous map from metric space (X, d) to metric space (Y, d') iff it is continuous at every $x \in X$.

Proof. Suppose f is continuous, then for every $x \in X$, every $U \subseteq Y$ that contains $f(x)$, $f^{-1}(U)$ is an open set in X that contains x and $f(f^{-1}(U)) \subseteq U$. Hence f is continuous at x. On the other hand, if for every $x \in X$, f is continuous at x, then for any open set $U \subseteq Y$, for any $x \in f^{-1}(U)$, $f(x) \in U$, hence there is an open set V_x in X such that $x \in V_x$ and $f(V_x) \subseteq U$, which implies $V_x \subseteq f^{-1}(U)$. This implies that $f^{-1}(U) = \bigcup_{x\in f^{-1}(U)} V_x$, hence $f^{-1}(U)$ must be open due to Theorem 3.2.8. □

Remark 3.3.6. It is evident from definition, that if $U \subseteq X$ is open, $x \in U$, then f is continuous at x iff $f|_U$ is continuous at x. Hence, if $X = \bigcup C$ and C is a set of open subsets in X, then $f : X \to Y$ is continuous iff $f|_U$ is continuous for all $U \in C$.

Example 3.3.7. A map $f : X \to Y$ is called **locally constant** if for every $x \in X$, there is an open set U containing x such that $f|_U$ is constant. Remark 3.3.6 implies that any locally constant function is continuous.

Definition 3.3.8. Let $f : X \to Y$ be a map, $A \subseteq X$, we say f is **continuous on** A iff f is continuous at every $a \in A$.

Theorem 3.3.9. $f : X \to Y$ is continuous at $x \in X$ iff for any $\epsilon > 0$, there is some $\delta > 0$, such that $f(B_X(x, \delta)) \subseteq B_Y(f(x), \epsilon)$.

Proof. Firstly we prove the "only if" part. Suppose f is continuous at x, the open ball $B_Y(f(x), \epsilon)$ is an open set (due to Theorem 3.2.6) that contains $f(x)$, hence there must be an open set $V \subseteq X$ containing x such that $f(V) \subseteq B_Y(f(x), \epsilon)$. The openness of V implies that there must be some $\delta > 0$ such that $B_X(x, \delta) \subseteq V$, hence $f(B_X(x, \delta)) \subseteq B_Y(f(x), \epsilon)$.

Now we prove the "if" part. Let U be an open set containing $f(x)$, there is some $\epsilon > 0$ such that $B_Y(f(x), \epsilon) \subseteq U$. By assumption there is some $\delta > 0$ such that $f(B_X(x, \delta)) \subseteq B_Y(f(x), \epsilon) \subseteq U$, and $B_X(x, \delta)$ is open due to Theorem 3.2.6. □

An immediate consequence of Theorem 3.3.9 and Theorem 3.3.5 is the following:

Theorem 3.3.10. A map f from (X, d) to (Y, d') is continuous iff for any $x \in X$, for any $\epsilon > 0$, there is some $\delta > 0$ such that for any $x' \in X$, if $d(x, x') < \delta$ then $d'(f(x), f(x')) < \epsilon$. □

Remark 3.3.11. A consequence of Theorem 3.3.10 is that if a map from (X, d) to (Y, d') is **Lipschitz** (or L-Lipschitz), i.e. there is some $L > 0$, such that for any $x, y \in X$, $d'(f(x), f(y)) \leq Ld(x, y)$, then it is continuous. In particular, if d and d' are two metrics on X, and there is some $c > 0$ such that $d' \leq cd$, then id_X as a map from (X, d) to (X, d') is continuous.

Example 3.3.12. The identity map as a map from $(\mathbb{R}^n, d_{Euclid})$ to $(\mathbb{R}^n, d_{sup})$ is 1-Lipschitz, its inverse is $\sqrt{n}$-Lipschitz, hence both are continuous.

Definition 3.3.13. If a bijection between two metric spaces and its inverse are both continuous we say it is a **homoemorphism**.

Example 3.3.14.

- Let (X, d) be a metric space, $x \in X$, then the function $d_x : X \to \mathbb{R}$ defined as $d_x(y) = d(x, y)$ is 1-Lipschitz hence continuous.
- Let (X, d) be a metric space, the function $(x, y) \mapsto d(x, y)$ from $(X \times X, d_{sup})$ to $\mathbb{R}$ is 2-Lipschitz, hence continuous.

3.3.3 Real Valued Continuous Functions

The continuity of maps to $\mathbb{R}$ can often be established via Theorem 3.3.10, Remark 3.3.11, or the following theorem:

Theorem 3.3.15. A function f from a metric space (X, d) to $\mathbb{R}$ is continuous iff $f^{-1}((-\infty, a))$ and $f^{-1}((a, \infty))$ are both open in X for all $a \in \mathbb{R}$.

Proof. By Theorem 3.2.16, any open subset of $\mathbb{R}$ can be written as a union of open balls of the form $(x-r, x+r)$ for some $x \in \mathbb{R}$, $r > 0$. Hence the Theorem follows from the fact that $(x-r, x+r) = (-\infty, x+r) \cap (x-r, \infty)$ and Theorem 3.2.8 and Theorem 3.2.12. $\square$

Example 3.3.16. Consider function $\sin : \mathbb{R} \to \mathbb{R}$.

$$|\sin(a) - \sin(b)| = |2\sin((a-b)/2)\cos((a+b)/2)| \leq |a-b|$$

Hence it is 1-Lipschitz hence continuous.

Example 3.3.17. We can also use these to show that sums, products and quotients of continuous functions are continuous. For example, if (X, d) is a metric space and $f, g : X \to \mathbb{R}$ are both continuous, we will show that so is fg:

- **Proof 1**: Let $x \in X$, for any $\epsilon > 0$, let $\epsilon' = \min(\frac{\epsilon}{|f(x)|+|g(x)|+1}, 1)$. Because f is continuous, there is some $\delta_1 > 0$ such that $f(B_X(x, \delta_1)) \subseteq (f(x) - \epsilon', f(x) + \epsilon')$. Similarly, there is some $\delta_2 > 0$ such that $g(B_X(x, \delta_2)) \subseteq (g(x) - \epsilon', g(x) + \epsilon')$. Let $\delta = \min(\delta_1, \delta_2) > 0$, then for any $x' \in X$, if $d(x, x') < \delta$, we have

$$|f(x)g(x) - f(x')g(x')|$$
$$\leq |f(x) - f(x')||g(x)| + |f(x)||g(x) - g(x')| + |f(x) - f(x')||g(x) - g(x')|$$
$$< \frac{\epsilon|g(x)|}{|f(x)| + |g(x)| + 1} + \frac{\epsilon|f(x)|}{|f(x)| + |g(x)| + 1} + \frac{\epsilon}{|f(x)| + |g(x)| + 1} = \epsilon$$

- **Proof 2**: Let $U_a = \{x \in X : f(x)g(x) > a\}$, $L_a = \{x \in X : f(x)g(x) < a\}$. We only need to show that for all $a \in \mathbb{R}$, U_a and L_a are both open.
  - When $a = 0$:

  $$U_0 = (f^{-1}((0,\infty)) \cap g^{-1}((0,\infty))) \cup (f^{-1}((-\infty,0)) \cap g^{-1}((-\infty,0)))$$
  $$L_0 = (f^{-1}((0,\infty)) \cap g^{-1}((-\infty,0))) \cup (f^{-1}((-\infty,0)) \cap g^{-1}((0,\infty)))$$

  - When $a > 0$:

  $$U_a = (\bigcup_{s>0}(f^{-1}((s,\infty)) \cap g^{-1}((a/s,\infty))))$$
  $$\cup(\bigcup_{s<0}(f^{-1}((-\infty,s)) \cap g^{-1}((-\infty,a/s))))$$
  $$L_a = L_0 \cup (\bigcup_{s>0}(f^{-1}((-s,s)) \cap g^{-1}((-a/s,a/s))))$$

  - The case when $a < 0$ is similar.

There is a key property for continuous functions on $\mathbb{R}$:

Theorem 3.3.18. (**Intermediate Value Property**) If $f : [a, b] \to \mathbb{R}$ is a continuous map, $f(a) < m < f(b)$ or $f(b) < m < f(a)$, then there is some $c \in (a, b)$ where $f(c) = m$.

Proof. We only prove the case when $f(a) < m < f(b)$ as the other case is completely analogous. Let $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(x) = \begin{cases} f(a) & x \leq a \\ f(b) & x \geq b \\ f(x) & a < x < b \end{cases}$$

Then g is also continuous, which can be shown either with Theorem 3.3.10 or by writing g as the composition of f and a 1-Lipschitz map $x \mapsto \min(b, \max(a, x))$.

Let $c = \sup(g^{-1}((-\infty, m]))$. Then because $g^{-1}((-\infty, m])$ is closed, Theorem 3.2.24 implies that $c \in g^{-1}((-\infty, m])$, hence $g(c) \leq m$. If $g(c) < m$, then $c \leq \sup(g^{-1}((-\infty, m))) \leq \sup(g^{-1}((-\infty, m])) = c$, hence $c = \sup(g^{-1}((-\infty, m)))$. But $g^{-1}((-\infty, m))$ is open, this contradicts with Theorem 3.2.24. □

3.4 Compactness

3.4.1 Definition and Basic Properties

Definition 3.4.1. A metric space X is called **compact**, if for any set C of open subsets that satisfies $\bigcup C = X$, there is a finite subset $C' \subseteq C$ such that $\bigcup C' = X$.

Remark 3.4.2. A set of open sets whose union is X is called an **open cover** of X. Hence, the definition above can be stated as "every open cover has a finite subcover".

Example 3.4.3. A discrete metric space (X, d_{disc}) is compact iff X is a finite set.

Theorem 3.4.4. A metric space (X, d) is compact iff any set C of open balls whose unions is X has a finite subset C' whose union is X.

Proof. The "only if" part is due to the fact that open balls are open. For the "if" part, let C_1 be any open cover of X, for every $x \in X$, there is some $U_x \in C_1$ such that $x \in U_x$, hence there is some $r_x > 0$ such that $B_X(x, r_x) \subseteq U_x$, and $\{B_X(x, r_x)\}$ is now a set of open balls whose union is X. By assumption, there are finitely points $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^n B_X(x_i, r_{x_i})$, hence $\{U_{x_1}, \ldots, U_{x_n}\}$ is a finite subcover of C_1. □

Example 3.4.5. Theorem 3.4.4 and Theorem 2.5.2 implies that finite closed intervals are compact.

Theorem 3.4.6. Any compact metric space has finite diameter.

Proof. Let x be a point in a compact metric space (X, d). Let $C = \{B_i : i \in \mathbb{N}\}$, where $B_i = B_X(x, i+1)$. Then $X = \bigcup C$ due to Theorem 2.3.1, hence there is a finite subset $C' = \{B_{i_1}, \ldots, B_{i_n}\} \subseteq C$ such that $X = \bigcup C' = B_{\max(i_1,\ldots,i_n)}$, hence $diam(x) \leq 2 + 2\max(i_1, \ldots, i_n)$ due to Remark 3.1.7. □

Theorem 3.4.7. If (X, d) is a metric space, $A \subseteq X$. Then if A under subspace metric is compact then A is closed in X.

Proof. We only need to show that for any $p \in X \backslash A$, there is some $r > 0$ such that $A \cap B_X(p, r) = \emptyset$. If this is not true, let $U_n = \{q \in A : d(q, p) > \frac{1}{n+1}\}$, then $C = \{U_n : n \in \mathbb{N}\}$ is an open cover of A that does not have a finite subcover. □

3.4.2 Compactness of images and subsets

Theorem 3.4.8. If $f : X \to Y$ is continuous, X is compact, then so is $f(X)$.

Proof. Let C be an open cover of $f(X)$. Because the map $f : X \to f(X)$ is continuous, $\{f^{-1}(U) : U \in C\}$ is an open cover of X. Compactness of X implies that there are $f^{-1}(U_1), \ldots, f^{-1}(U_n)$ whose union is X, hence $f(X) = \bigcup_{i=1}^n U_i$. □

As an application we have:

Theorem 3.4.9. Let (X, d) be a non-empty compact metric space, $f : X \to \mathbb{R}$ a continuous function. Then f reaches its maximum and minimum, i.e. there are $x_m, x_M \in X$ such that for any $x' \in X$, $f(x_m) \leq f(y) \leq f(x_M)$.

Proof. By Theorem 3.4.8, $f(X)$ is compact. By Theorem 3.4.6 it has supremum and infimum. By Theorem 3.4.7 it is closed, hence by Theorem 3.2.24 $\sup(f(X)) \in f(X)$, $\inf(f(X)) \in f(X)$. □

Theorem 3.4.10. A closed subset of a compact metric space is compact.

Proof. Let V be a closed subset of a compact metric space (X, d), C an open cover of V. For every $U \in C$, by Theorem 3.2.17, let U' be an open set in X such that $U' \cap V = U$. Then $\{U' : U \in C\} \cup \{X \backslash V\}$ is an open cover of X, by compactness it has a finite subcover. The non-empty intersection of elements in this finite subcover with V gives us a finite subcover of C. □

3.4.3 Compactness of subsets of $\mathbb{R}^n$

Theorem 3.4.11. Let (X, d) and (Y, d') be two compact metric spaces, then $(X \times Y, d_{sup})$ as defined in Theorem 3.1.10 is also compact.

Proof. It is easy to see that

$$B_{X\times Y}((a, b), r) = B_X(a, r) \times B_X(b, r)$$

Suppose C is an open cover of $X \times Y$. For every $(x, y) \in X \times Y$, let $U_{x,y}$ be an element in C that contains (x, y), and $r_{x,y} > 0$ a number such that

$B_{X\times Y}((x,y),r_{x,y}) \subseteq U_{x,u} \in C$. Hence, for every $x \in X$, the set $\{B_Y(y,r_{x,y}) : y \in Y\}$ is an open cover of Y. Hence there must be finitely many points $y_{x,1},\dots,y_{x,n_x}$ such that $Y = \bigcup_{i=1}^{n_x} B_Y(y_{x,i}, r_{x,y_{x,i}})$. Let $r_x = \min\{r_{x,y_{x,i}} : i = 1,\dots,n_x\}$. Then

$$B_X(x,r_x) \times Y \subseteq \bigcup_{i=1}^{n_x} B_{X\times Y}((x,y_{x,i}), r_{x,y_{x,i}})$$

Because $r_x > 0$ for all x, $X = \bigcup_{x\in X} B_X(x,r_x)$. Compactness of X implies that there must be finitely many $x_1,\dots,x_m \in X$ such that $X = \bigcup_{j=1}^{m} B_X(x_j, r_{x_j})$. Hence

$$X \times Y = \bigcup_{j=1}^{m}(\bigcup_{i=1}^{n_{x_j}} B_{X\times Y}((x_j, y_{x_j,i}), r_{x_j,y_{x_j,i}})) \subseteq \bigcup_{j=1}^{m}\bigcup_{i=1}^{n_{x_j}} U_{x_j,y_{x_j,i}}$$

The set $\{U_{x_j,y_{x_j,i}}\} \subseteq C$ is a finite cover. □

Theorem 3.4.12. Let $a_i < b_i$ be n pairs of real numbers, the set $\prod_{i=1}^{n}[a_i,b_i] \subseteq \mathbb{R}^n$ is compact.

Proof. Example 3.4.5 and Theorem 3.4.11 shows that this set as a subspace of $(\mathbb{R}^n, d_{sup})$ is compact. Example 3.3.12 and Theorem 3.4.8 shows that it is compact as a subspace of $\mathbb{R}^n$ with Euclidean metric. □

Combining Theorem 3.4.6, Theorem 3.4.7, Theorem 3.4.12 and Theorem 3.4.10 we have:

Theorem 3.4.13. A subset of $\mathbb{R}^n$ is compact iff it is bounded and closed.

□

4 Sequential Limit

4.1 Definition and Basic Properties

Definition 4.1.1. Let (X, d) be a metric space.

- A **Sequence** on X is a function from $\mathbb{N}$ to X. We often write the parameter in the function in the subscript, and denote a sequence as something like $\{a_n\}$.
- Let $\{a_n\}$ be a sequence. We say that b is the **limit** of $\{a_n\}$, denoted as $\lim_{n\to\infty} a_n = b$, iff for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n > N$ then $d(a_n, b) < \epsilon$.
- If a sequence has a limit we say it **converges**. Otherwise we say it **diverges**.

Example 4.1.2.

- The **constant sequence** $a_n = c$ has limit c.
- In $\mathbb{R}$, the sequence $a_n = 2^{-n}$ has limit 0.

Theorem 4.1.3. Let (X, d) be a metric space. If a sequence $\{a_n\}$ on X has a limit, then the limit must be unique.

Proof. Suppose $\lim_{n\to\infty} a_n = b$, $\lim_{n\to\infty} a_n = c$. For any $\epsilon > 0$, there is some N such that if $n > N$, $d(a_n, b) < \epsilon/2$, There is also some N' such that if $n > N'$, $d(a_n, c) < \epsilon/2$. Let $N'' = \max(N, N') + 1$, then $d(b, c) \leq d(b, a_{N''}) + d(c, a_{N''}) < \epsilon$, hence $d(b, c) = 0$ because the only real number in $[0, 1)$ smaller than all positive real numbers is 0. Hence $b = c$. □

Theorem 4.1.4. If sequences $\{a_n\}$ and $\{b_n\}$ on a metric space (X, d) differ at only finitely many places, then $\{a_n\}$ converges iff $\{b_n\}$ converges, and when they converge they have the same limit.

Proof. Let M be the largest natural number where $a_M \neq b_M$. Suppose $\lim_{n\to\infty} a_n = c$, then for any $\epsilon > 0$, there is some N such that $n > N$ implies $d(a_n, c) < \epsilon$. Hence for any $n > \max(N, M)$, $d(b_n, c) < \epsilon$. Hence $\lim_{n\to\infty} b_n = c$. □

4.2 Limit and Continuity

Theorem 4.2.1. Let (X, d) be a metric space, $A \subseteq X$ a non-empty subset. Then

- A is open iff for any sequence $\{a_n\}$ on X, if $\lim_{n\to\infty} a_n \in A$ then $a_n \in A$ for all but finitely many n. (i.e. there is some $N \in \mathbb{N}$ such that $n > N$ implies $a_n \in A$.)
- A is closed iff for any sequence $\{a_n\}$ on X, if $a_n \in A$ for all n, $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} a_n \in A$.

Proof.

- Suppose A is open. If $\lim_{n\to\infty} a_n = m \in A$, there is some $\epsilon > 0$ such that $B_X(m,\epsilon) \subseteq A$. Hence the existence of such N follows from the definition of limit. On the other hand, if A is not open, then there is some $x \in A$ where for any $r > 0$, $B_X(x,r) \not\subseteq A$. Let $a_n \in B_X(x, 2^{-n})\backslash A$, then $a_n \notin A$ for all n and $\lim_{n\to\infty} a_n \in A$.
- Suppose A is closed, then $X\backslash A$ is open, hence any sequence with limit in $X\backslash A$ must be in $X\backslash A$ for infinitely many n. As a consequence, if $a_n \in A$ for all n then $\lim_{n\to\infty} a_n$ can not be in $X\backslash A$. On the other hand, if A is not closed, then there is some $x \notin A$ where for any $r > 0$, $B_X(x,r) \cap A \neq \emptyset$. Let $a_n \in B_X(x, 2^{-n}) \cap A$, then $\{a_n\}$ is a sequence in A whose limit is not in A.

□

Remark 4.2.2. Let x be a point in a metric space (X, d). If there is some $r > 0$ such that $B_X(x,r) = \{x\}$, we call x an **isolated point**. Theorem 4.2.1 implies that if x is an isolated point then any sequence $\{a_n\}$ convergent to x must have $a_n = x$ for all but finitely many n. Also, any function from X to another metric space Y must be locally constant at x hence continuous at x (Example 3.3.7). This is an example for the Theorem below.

Theorem 4.2.3. Let (X, d) and (Y, d') be two metric spaces, $f : X \to Y$ a map, then the followings are equivalent:

1. f is continuous at $x \in X$.
2. $\{a_n\}$ is a sequence on X. If $\lim_{n\to\infty} a_n = x$, then $\lim_{n\to\infty} f(a_n) = f(x)$.
3. $\{a_n\}$ is a sequence on X. If $\lim_{n\to\infty} a_n = x$, then $\{f(a_n)\}$ converges.

Proof. **1 implies 2**: By Theorem 3.3.9, for any $\epsilon > 0$, there is $\delta > 0$ such that $f(B_X(x,\delta)) \subseteq B_Y(f(x),\epsilon)$. Let $N \in \mathbb{N}$ be such that $n > N$ implies $d(a_n, x) < \delta$, the $d'(f(a_n), f(x)) < \epsilon$, this shows that $\lim_{n\to\infty} f(a_n) = f(x)$.

2 implies 3: This is obvious.

3 implies 1: Suppose f is not continuous at x, then there is $\epsilon > 0$ such that for any $n \in \mathbb{N}$, $f(B_X(x, 2^{-n})) \not\subseteq B_Y(f(x), \epsilon)$. For every $n \in \mathbb{N}$, pick $b_n \in B_X(x, 2^{-n})\backslash f^{-1}(B_Y(f(x), \epsilon))$, and let

$$a_n = \begin{cases} b_n & n \text{ is odd} \\ x & n \text{ is even} \end{cases}$$

Then $\lim_{n\to\infty} a_n = x$ while $\lim_{n\to\infty} f(a_n)$ does not exist. □

Remark 4.2.4. Theorem 3.3.5 and Theorem 4.2.3 imply that f is continuous iff it sends convergent sequences to convergent sequences.

Remark 4.2.5. The inclusion map is continuous, hence if a sequence $\{a_n\}$ on $Y \subseteq X$ converges and has a limit $b \in Y$, then its limit in X is also b.

Theorem 4.2.6. Let $P = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$, metric is the Euclidean metric. Let (X, d) be a metric space, $\{a_n\}$ a sequence on X, $b \in X$. Define function $f_{a,b} : P \to X$ as $f_{a,b}(2^{-n}) = a_n$, $f_{a,b}(0) = b$. Then the followings are equivalent:

1. $\lim_{n\to\infty} a_n = b$
2. The function $f_{a,b}$ is continuous.
3. The function $f_{a,b}$ is continuous at 0.

Proof. The equivalence of 2 and 3 is due to the fact that 0 is the only non-isolated point in P (see Remark 4.2.2). 3 implies 1 follows immediately from Theorem 4.2.3. To show 1 implies 3, for any $\epsilon > 0$, let $N \in \mathbb{N}$ such that $n > N$ implies $d(a_n, b) < \epsilon$, hence $f_{a,b}$ sent $B_P(0, 2^{-N}) = \{0\} \cup \{2^{-n} : n > N\}$ to $B_X(b, \epsilon)$, hence continuous at 0. □

Remark 4.2.7. The set P above is compact due to Theorem 3.4.13. Hence, the range of any convergent sequence is bounded.

Remark 4.2.8. A **subsequence** is the composition between a sequence $\{a_n\}$ and an increasing injection $i : \mathbb{N} \to \mathbb{N}$, denoted as $\{a_{i_n}\}$. Theorem 4.2.3 and Theorem 4.2.6 implies that if a sequence converges so is any of its subsequences, and the limit of the subsequence is the same as the limit of the original sequence.

Definition 4.2.9. If f is a map from a metric space (X, d) to metric space (Y, d'), $x \in X$ is not an isolated point. We say the **limit of f at x is b**, denoted as $\lim_{t\to x} f(t) = b$, iff the function

$$g(t) = \begin{cases} f(t) & t \neq x \\ b & t = x \end{cases}$$

is continuous at x.

Remark 4.2.10. When x is a non-isolated point in X, there is a sequence $\{a_n\}$ convergent to x such that $a_n \neq x$ for all n. Hence $\lim_{t\to x} f(t)$ is unique when it exists.

4.3 Cauchy Sequences

Definition 4.3.1. A sequence $\{a_n\}$ on a metric space (X, d) is called a **Cauchy sequence** if for any $\epsilon > 0$, there is some $N \in \mathbb{N}$, such that for any $m, n > N$, $d(a_m, a_n) < \epsilon$.

Theorem 4.3.2. Any convergent sequence is a Cauchy sequence.

Proof. Suppose $\lim_{n\to\infty} a_n = b$. For any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $n > N$ implies $d(a_n, b) < \epsilon/2$. Hence for any $m, n > N$, $d(a_m, a_n) \leq d(a_m, b) + d(a_n, b) < \epsilon$. □

Theorem 4.3.3. A closed subset of a complete metric space is complete.

Proof. Let (X, d) be a complete metric space, $V \subseteq X$ closed. Then for any Cauchy seqeuence $\{a_n\}$ on V, completeness of X implies that it converges to some $b \in X$, and Theorem 4.2.1 implies that $b \in V$. □

Theorem 4.3.4. Let (X, d) be a metric space, if $A \subseteq X$ is complete under subspace metric, then A is closed.

Proof. Suppose not, Theorem 4.2.1 implies that there will be a sequence $\{a_n\}$ on A which converges to some $b \in X \backslash A$. Theorem 4.3.2 implies that $\{a_n\}$ is a Cauchy sequence, hence A is not complete. □

Theorem 4.3.5. Let (X, d) be a metric space. If there is some $r > 0$ such that any closed ball with radius r in X is compact, then (X, d) is complete. In particular, any compact metric space is complete.

To show this we first need the following Lemma:

Lemma 4.3.6. Let $\{a_n\}$ be a Cauchy sequence on metric space (X, d), $r > 0$, then there is some $x \in X$ such that for all but finitely many n, a_n lies in the closed ball centered at x with radius r.

Proof. Let $N \in \mathbb{N}$ satisfies that for all $m, n > N$, $d(a_m, a_n) < r$, then for all $n > N$, $a_n \in B_X(a_{N+1}, r)$. □

Lemma 4.3.7. Let $\{a_n\}$ be a sequence on a compact metric space X, then there is some $x \in X$ such that for any $r > 0$, any $m \in \mathbb{N}$, there is some $n > m$ such that $a_n \in B_X(x, r)$.

Proof. If this is not true, then for every $x \in X$, there is some $r_x > 0$, $m_x \in \mathbb{N}$, such that if $n > m_x$ then $a_n \notin B_X(x, r_x)$. $X = \bigcup_{x \in X} B_X(x, r_x)$, hence compactness of X implies that there must be $x_1, \ldots, x_k \in X$ such that $X = \bigcup_{i=1}^{k} B_X(x_i, r_{x_i})$. Let $n = \max\{m_{x_1}, \ldots, m_{x_k}\} + 1$, then a_n can not be in X, a contradiction. □

Proof of Theorem 4.3.5. By Lemma 4.3.6 we can assume without loss of generality that $\{a_n\}$ lies on a compact closed ball $B \subseteq X$. Now apply Lemma 4.3.7 we can get a point $x \in B$ that satisfies the conclusion of Lemma 4.3.7. We will now show that $\lim_{n \to \infty} a_n = x$. For any $\epsilon > 0$, let N be such that $m, n > N$ implies $d(a_m, a_n) < \epsilon/2$. Let $n' > N$ be such that $d(a_{n'}, x) < \epsilon/2$, then for any $n > N$, $d(a_n, x) \leq d(a_n, a_{n'}) + d(a_{n'}, x) < \epsilon$. □

Example 4.3.8.

- Theorem 4.3.5 implies that any discrete metric space is complete.
- Theorem 4.3.5 implies that $\mathbb{R}$ is complete.
- Theorem 4.3.4 implies that $\mathbb{Q}$ or $(0, 1)$ under Euclidean metric are not complete.

Remark 4.3.9. Unlike compactness, completeness is not preserved by homeomorphisms. For example $(0,1)$ and $\mathbb{R}$ are homeomorphic, the former isn't complete while the latter is.

Theorem 4.3.10. Let (X,d) be a complete metric space, S a non-empty set. Then $(BF(S,X), d_{sup})$ is also complete.

Proof. Let $\{a_n\}$ be a Cauchy sequence in $BF(S,X)$. For any $s \in X$, any $m,n \in \mathbb{N}$, $d(a_m(s), a_n(s)) \leq d_{sup}(a_m, a_n)$, hence $\{a_n(s)\}$ is a Cauchy sequence on X. By completeness of X there is some $b_s \in X$ where $\lim_{n\to\infty} a_n(s) = b_s$. Now define $b : S \to X$ as $s \mapsto b_s$, we need to show $b \in BF(S,X)$ and $\lim_{n\to\infty} a_n = b$.

Step 1: We first show that $b \in BF(S,X)$. Let $s_0 \in S$. Let $N \in \mathbb{N}$ satisfies that for any $m,n > N$, $d_{sup}(a_m, a_n) < 1$. $a_{N+1} \in BF(S,X)$ implies that $diam(a_{N+1}(S)) = D < \infty$, hence $a_{N+1}(S) \subseteq B_X(a_{N+1}(s_0), D+1)$. For any $s \in S$, any $n > N$, $\{a_n(s)\}$ lies in the closed ball with radius 1 centered at $a_{N+1}(s)$, hence $d(b(s), a_{N+1}(s)) \leq 1$, hence $b(S) \subseteq B_X(a_{N+1}(s_0), D+2)$, $diam(b(S)) \leq 2D + 4 < \infty$.

Step 2: We now show that $\lim_{n\to\infty} a_n = b$. Let $\epsilon > 0$, pick $N \in \mathbb{N}$ such that when $n,m > N$, $d_{sup}(a_m, a_n) < \epsilon/3$. Now for any $s \in S$, when $n > N$, $a_n(s)$ lies in the closed ball centered at $a_{N+1}(s)$ with radius $\epsilon/3$, hence by Theorem 4.2.1, $b(s) = \lim_{n\to\infty} a_n(s)$ is in this closed ball with radius $\epsilon/3$ centered at $a_{N+1}(s)$ as well. Hence any $n > N$, $d(a_n(s), b(s)) \leq d(a_n(s), a_{N+1}(s)) + d(a_{N+1}(s), b(s)) < 2\epsilon/3$, hence $d_{sup}(a_n, b) \leq 2\epsilon/3 < \epsilon$. This means that $\lim_{n\to\infty} a_n = b$. □

Remark 4.3.11. When S is an infinite set and $X = \mathbb{R}$, $BF(S,X)$ is complete but does not satisfy the assumption of Theorem 4.3.5. In particular, for any $r > 0$, the closed ball B_r centered at 0 with radius r is not compact.

To show this, suppose B_r is compact, let $i : \mathbb{N} \to S$ be an injection, consider sequence $\{f_n\}$ on B_r defined as

$$f_n(s) = \begin{cases} r & s = i(n) \\ 0 & s \neq i(n) \end{cases}$$

Then Lemma 4.3.7 implies that there must be some $g \in B_r$ such that for all but finitely many n, $d_{sup}(g, f_n) < r/3$, which is impossible.

4.4 Uniform Convergence

Theorem 4.4.1. Let (X,d), (Y,d') be two metric spaces, $BF(X,Y)$ be the set of bounded functions (i.e. functions with bounded image). For any $x \in X$, let $C_x = \{f \in BF(X,Y) : f \text{ is continuous at } x\}$, then C_x is closed in $(BF(X,Y), d_{sup})$.

Proof. Suppose $f \in BF(X,Y)\backslash C_x$, then there is some $\epsilon > 0$, such that for any $r > 0$, there is some $x'_r \in (B_X(x,r))$ such that $d'(f(x'_r), f(x)) \geq \epsilon$. Let $g \in B_{BF(X,Y)}(f, \epsilon/3)$, then $d'(g(x'_r), g(x)) \geq \epsilon/3$, hence $B_{BF(X,Y)}(f, \epsilon/3) \cap C_x = \emptyset$. □

Remark 4.4.2. Let $BC(X,Y)$ be the set of bounded continuous functions from X to Y, then $BC(X,Y) = \bigcup_{x \in X} C_x$ hence is closed in $BF(X,Y)$.

Remark 4.4.3. By Theorem 4.4.1 and Remark 4.4.2, as well as Theorem 4.3.10 and Theorem 4.3.3, C_x and $BC(X,Y)$ are both complete under d_{sup}.

Definition 4.4.4. Let (X,d) be a metric space, $S \neq \emptyset$. A sequence of maps $f_n : S \to X$ are said to **converges uniformly to a map** $f : S \to X$, if for any $\epsilon > 0$, there is some $N \in \mathbb{N}$, such that for any natural number $n > N$, any $s \in S$, $d(f_n(s), f(s)) < \epsilon$. It is said to be **uniformly Cauchy**, if for any $\epsilon > 0$, there is some $N \in \mathbb{N}$, such that for any $s \in S$, any $m, n > N$, $d(f_n(s), f_m(s)) < \epsilon$.

Lemma 4.4.5. Let (X,d) be a metric space, $d_1 = \min(d,1)$, S a non-empty set, $\{x_n\}$ a sequence in X, and $\{f_n\}$ a sequence of functions from S to X. Then:

1. The sequence $\{x_n\}$ is Cauchy in (X,d) iff it is Cauchy in (X,d_1).
2. The sequence $\{x_n\}$ converges to $b \in X$ under metric d iff it converges to b under metric d_1.
3. The sequence $\{f_n\}$ is uniformly Cauchy in (X,d) iff it is Cauchy in (X,d_1).
4. The sequence $\{f_n\}$ converges to $f \in Map(S,X)$ under metric d iff it converges to f under metric d_1.

Proof. To prove 1, if $\{x_n\}$ is Cauchy in (X,d), for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $m, n > N$ implies $d(x_n, x_m) < \epsilon$. Hence, as long as $m, n > N$, $d_1(x_m, x_n) \leq d(x_m, x_n) < \epsilon$. On the other hand, if $\{a_n\}$ is Cauchy under (X, d_1), for any $\epsilon > 0$, let $N \in \mathbb{N}$ such that when $m, n > N$, $d_1(x_m, x_n) < \min(1, \epsilon)$. Then $d(x_m, x_n) < \min(1, \epsilon) \leq \epsilon$.

The proofs of the other statements are all analogous. □

Remark 4.4.6. Part 2 of Lemma 4.4.5 implies that the identity map is an homeomorphism from (X,d) to $(X, \min(d,1))$.

Theorem 4.4.7. If (X,d) is a complete metric space, $f_n : S \to X$ a sequence of functions, then if $\{f_n\}$ is uniformly Cauchy then it converges uniformly to some $f : S \to X$.

Proof. If $\{f_n\}$ is uniformly Cauchy, it is uniformly Cauchy as a sequence of maps from S to $(X, \min(d,1))$, hence by Lemma 4.4.5 a Cauchy sequence in $BF(S,X)$ where the metric on X is $\min(d,1)$. By Theorem 4.3.10 it converges uniformly to some function f from S to X. Apply Lemma 4.4.5 again we get the conclusion of this theorem. □

Theorem 4.4.8. (Uniform Convergence Theorem) If (X, d) and (Y, d') are two metric spaces, $f_n : X \to Y$ converges uniformly to some $f : X \to Y$.

1. If f_n are all continuous at some $x \in X$, then so is f.
2. If f_n are all continuous, then f is continuous as well.

Proof. By Lemma 4.4.5, $\{f_n\}$ converges to f under d_{sup} as bounded functions from X to $(Y, \min(1, d'))$. To prove the first statement, suppose all f_n are continuous at x as maps from (X, d) to (Y, d'), then by Remark 4.4.6, they are continuous at x as maps from (X, d) to $(Y, \min(1, d'))$. By Theorem 4.4.1 and Theorem 4.2.1, f is continuous at x as a map from (X, d) to $(Y, \min(1, d'))$. By Remark 4.4.6 it is continuous at x as a map from (X, d) to $(Y, \min(1, d'))$. The proof of the second statement is similar, just replace Theorem 4.4.1 above with Remark 4.4.2. □

Example 4.4.9. Consider real-valued functions on $[0, 1]$ $f_n : x \mapsto x^n$. Here we set $x^0 = 1$ for all x, and $g(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$. Then for any $x \in [0, 1]$, $\lim_{n\to\infty} f_n(x) = g(x)$, but f_n does not converge uniformly to g on $[0, 1]$. However, for any $0 < r < 1$, $f_n|_{[0,r]}$ does converge uniformly to $g|_{[0,r]}$.

Example 4.4.10. As an example of Theorem 4.4.8, let $\{a_{m,n}\}$ be a map from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{R}$. If $\lim_{n\to\infty} a_{m,n} = a_m$, and for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for $m, m' > N$, for any n, $|a_{m,n} - a_{m',n}| < \epsilon$, then

$$\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} = \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n}$$

5 Sequences and Series in $\mathbb{R}$

Now we will apply techniques developed in previous lectures to study the convergence of sequences and series on $\mathbb{R}$.

5.1 Monotone convergence, upper and lower limit

Theorem 5.1.1. (Monotone Convergence Theorem) Let $\{x_n\}$ be a sequence on $\mathbb{R}$.

1. If $x_{n+1} \geq x_n$ for all n, and there is some $M \in \mathbb{R}$ such that $x_n \leq M$ for all n, then $\{x_n\}$ converges, and $\lim_{n\to\infty} x_n \geq x_n$ for all $n \in \mathbb{N}$.

2. If $x_{n+1} \leq x_n$ for all n, and there is some $m \in \mathbb{R}$ such that $m \leq x_n$ for all n, then $\{x_n\}$ converges, and $\lim_{n\to\infty} x_n \leq x_n$ for all $n \in \mathbb{N}$.

Proof. We only need to prove Part 1, and Part 2 follows from Part 1 if one replaces $\{x_n\}$ with $\{-x_n\}$.

By assumption, the set $X = \{x_n : n \in \mathbb{N}\}$ has an upper bound hence has the smallest upper bound, denote it as x. Now for any $\epsilon > 0$, if $X \cap (x - \epsilon, x] = \emptyset$, then $x - \epsilon$ is another upper bound of X, which contradicts the minimality of x. Hence $X \cap (x - \epsilon, x] \neq \emptyset$, in other words, there is some $N \in \mathbb{N}$ such that $x - \epsilon < x_N \leq x$. Hence for any $n > N$, $x - \epsilon < x_N \leq x_n \leq x$, $|x_n - x| < \epsilon$. This shows that $\lim_{n\to\infty} x_n = x$. □

Definition 5.1.2. Let $\{x_n\}$ be a sequence on $\mathbb{R}$. Suppose $\{x_n : n \in \mathbb{N}\}$ has upper and lower bound in $\mathbb{R}$. Then:

- The sequence $y_n = \sup(\{x_j : j \in \mathbb{N}, j \geq n\})$ is decreasing and has a lower bound, hence has a limit, denoted as $\limsup_{n\to\infty} x_n$, called the **upper limit**.

- The sequence $z_n = \inf(\{x_j : j \in \mathbb{N}, j \geq n\})$ is increasing and has an upper bound, hence has a limit, denoted as $\liminf_{n\to\infty} x_n$, called the **lower limit**.

Example 5.1.3. Let $x_n = (-1)^n$, then

$$\limsup_{n\to\infty} x_n = 1$$

$$\liminf_{n\to\infty} x_n = -1$$

Example 5.1.4. Let $x_n = \sin(n)$. Then we can show that $\sup\{\sin(j) : j \geq n\} = 1$ for all $n \in \mathbb{N}$, which implies that $\limsup_{n\to\infty} \sin(n) = 1$.

Because $\sin(n) \leq 1$, for all n, $\sup\{\sin(j) : j \geq n\} \leq 1$. To show that this supremum equals 1, we only need to show that for any $\epsilon > 0$, there is some $k \in \mathbb{N}$, some $j \geq n$, such that $|j - (2k\pi + \frac{1}{2})\pi| < \epsilon$. Pick $M \in \mathbb{N}$

such that $\epsilon > 1/M$. Now consider the sequence $\{2l\pi - \lfloor 2l\pi \rfloor\}$. Because π is irrational, the terms of these sequence are all distinct, and they all lie in a finite interval $[0, 1)$, hence pigeon hole principle implies that there must be $l < l'$ such that $|(2l\pi - \lfloor 2l\pi \rfloor) - (2l'\pi - \lfloor 2l'\pi \rfloor)| < \frac{1}{M}$, which implies that $2\pi(l' - l) - \lfloor 2\pi(l' - l) \rfloor < \frac{1}{M}$. Now consider the sequence $\{(2n + 2k(l' - l) + \frac{1}{2})\pi - \lfloor (2n + 2k(l' - l) + \frac{1}{2})\pi \rfloor\}$, there must be some natural number $k \leq M$ such that $(2n + 2k(l' - l) + \frac{1}{2})\pi - \lfloor (2n + 2k(l' - l) + \frac{1}{2})\pi \rfloor \leq \frac{1}{M}$, and we can then let $j = \lfloor (2n + 2k(l' - l) + \frac{1}{2})\pi \rfloor$.

Remark 5.1.5. A bounded sequence $\{x_n\}$ in $\mathbb{R}$ converges iff $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.

5.2 Series

Definition 5.2.1. Let $\{a_n\}$ be a sequence in $\mathbb{R}$.

- The sequence $s_n = \sum_{i=0}^{n} a_n$ is called the **partial sum**.
- If $\{s_n\}$ converges, we denote the limit as the **sum of infinite series** $\sum_{i=0}^{\infty} a_i$, and say the series **converges**. Otherwise we say the series **diverges**.

The fact that $\mathbb{R}$ is complete implies the following:

Theorem 5.2.2. A sequence $\{x_n\}$ converges iff it is a Cauchy sequence, i.e. for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for any $m, n > N$, $|x_m - x_n| < \epsilon$.

□

Apply Theorem 5.2.2 to the sequence of partial sums s_n, we get:

Theorem 5.2.3. A series $\sum_{n=0}^{\infty} a_n$ converges iff for any $\epsilon > 0$, there is $N \in \mathbb{N}$, such that for any $m, m' > N$, $m \leq m'$, $|\sum_{i=m}^{m'} a_i| < \epsilon$.

□

Remark 5.2.4. As a consequence of Theorem 5.2.3, if $\sum_{n=0} \infty a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

5.2.1 Comparison test

Another consequence of Theorem 5.2.3 is the following **comparison test for convergence of series**:

Theorem 5.2.5. Suppose $|a_n| \leq b_n$, $\sum_{n=0}^{\infty} b_n$ converges, then so is $\sum_{n=0}^{\infty} a_n$, and $|\sum_{n=0}^{\infty} a_n| \leq \sum_{n=0}^{\infty} b_n$.

Proof. Because $|a_n| \leq b_n$, $b_n \geq 0$ for all n. By Theorem 5.2.3, the fact that $\sum_{n=0}^{\infty} b_n$ converges implies that for any $\epsilon > 0$, for any $m' \geq m > N$, $\sum_{i=m}^{m'} b_i = |\sum_{i=m}^{m'} b_i| < \epsilon$, hence for any $m' \geq m > N$, $|\sum_{i=m}^{m'} a_i| \leq \sum_{i=m}^{m'} |a_i| \leq$

$\sum_{i=m}^{m'} b_i < \epsilon$, hence $\sum_{n=0}^{\infty} a_n$ converges by Theorem 5.2.3.

For any $n \in \mathbb{N}$, any $m > n$, $\sum_{i=0}^{m} b_i \geq \sum_{i=0}^{n} b_i$. Because the interval $[\sum_{i=0}^{n} b_i, \infty)$ is closed and the sequence $\{\sum_{i=0}^{m} b_i\}_{m\in\mathbb{N}}$ is on $[\sum_{i=0}^{n} b_i, \infty)$ for all but finitely many m, its limit $\sum_{m=0}^{\infty} b_m = \lim_{m\to\infty} \sum_{i=0}^{m} b_m \geq \sum_{i=0}^{n} b_i$. Because $|\sum_{i=0}^{m} a_i| \leq \sum_{i=0}^{m} b_i \leq \sum_{m=0}^{\infty} b_m$, the convergent sequence $\{\sum_{i=0}^{m} a_i\}_{m\in\mathbb{N}}$ lies on closed set $[-\sum_{n=0}^{\infty} b_n, \sum_{n=0}^{\infty} b_n]$, hence its limit $\sum_{n=0}^{\infty} a_n$ also lies in this closed set. □

As an application, we have:

Theorem 5.2.6. If $\sum_{n=0}^{\infty} |a_n|$ converges then so does $\sum_{n=0}^{\infty} a_n$. In this case we call the series **converges absolutely**.

□

A property of absolute convergent series is that the order of summation can be changed arbitrarily. More precisely,

Theorem 5.2.7. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection, suppose $\sum_{n=0}^{\infty} |a_n|$ converges, then so is $\sum_{n=0}^{\infty} a_{\sigma(n)}$, and

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{\sigma(n)}$$

Proof. For any $\epsilon > 0$, by Theorem 5.2.3, pick N' such that for any $m \geq n > N'$, $\sum_{i=n}^{n} |a_i| < \epsilon/3$. Let

$$N = \max\{\sigma^{-1}(0), \ldots, \sigma^{-1}(N')\}$$

Then for any $m \geq n > N$,

$$\left|\sum_{i=n}^{m} a_{\sigma(i)}\right| \leq \sum_{i=n}^{m} |a_{\sigma(i)}| \leq \sum_{i=\min\{\sigma(n),\ldots,\sigma(m)\}}^{\max\{\sigma(n),\ldots,\sigma(m)\}} |a_i| < \epsilon/3$$

The last $<$ is because when $i > N$, $\sigma(i) > N'$. This shows that $\sum_{n=0}^{\infty} a_{\sigma(n)}$ converges.

Now we show that these two series have the same value. Let $S = \sum_{n=0}^{\infty} a_n$. For any $n > \max(N, N')$,

$$\left|\sum_{i=0}^{n} a_{\sigma(i)} - \sum_{i=0}^{N'} a_i\right| \leq \sum_{i>N', \sigma^{-1}(i)\leq n} |a_i| < \epsilon/3$$

On the other hand, for all $n > N$,

$$\left|\sum_{i=0}^{n} a_i - \sum_{i=0}^{N'} a_i\right| < \epsilon/3$$

So the limit of $\{\sum_{i=0}^{n} a_i\}$ must also be in the closed ball centered at $\sum_{i=0}^{N} a_i$ with radius $\epsilon/3$, in other words

$$\left|\sum_{i=0}^{N'} a_i - S\right| \leq \epsilon/3$$

Hence by triangle inequality,

$$\left|\sum_{i=0}^{n} a_{\sigma(i)} - S\right| < \epsilon$$

This shows that $\sum_{n=0}^{\infty} a_{\sigma(n)} = S$. □

Remark 5.2.8. If $\sum_{n=0}^{\infty} a_n$ converges while $\sum_{n=0}^{\infty} |a_n|$ diverges, $\sum_{n=0}^{\infty} a_{\sigma(n)}$ can take any real value, or diverges to infinity or negative infinity.

5.2.2 Power Series

As another application of Theorem 5.2.5, we have:

Theorem 5.2.9. Let $\{a_n\}$ be a sequence on $\mathbb{R}$. Suppose $M = \limsup_{n\to\infty} |a_n|^{1/n}$. Then:

1. If $|x|M > 1$ then $\sum_{n=0}^{\infty} a_n x^n$ diverges.
2. If $|x|M < 1$ then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.
3. If $r > 0$ and $rM < 1$, then $\sum_{n=0}^{\infty} a_n x^n$ converges on $[-r, r]$ uniformly.

Proof. 1. For any $N \in \mathbb{N}$, by definition of $\limsup$, $\sup\{|a_j|^{1/j} : j > N\} \geq M > \frac{1}{|x|}$, hence there is some $n > N$ such that $|a_n|^{1/n} > \frac{1}{|x|}$, hence

$$|a_n x^n| = (|a_n|^{1/n}|x|)^n > 1$$

This shows that $a_n x^n$ does not converge to 0. By Remark 5.2.4 this implies $\sum_{n=0}^{\infty} a_n x^n$ diverges.

2. The case when $x = 0$ is obvious, so we now assume $x \neq 0$. By definiton of $\limsup$, there is some $N \in \mathbb{N}$ such that

$$\sup\{|a_n|^{1/n} : n \geq N\} < \frac{1}{2}(M + \frac{1}{|x|})$$

Let

$$b_n = \begin{cases} |a_n x^n| & n \leq N \\ (\frac{1}{2}(M|x| + 1))^n & n > N \end{cases}$$

Then $|a_n x^n| \leq b_n$ for all n.

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{N} |a_n||x|^n + \lim_{l\to\infty} \sum_{n=N+1}^{l} (\frac{1}{2}(M|x|+1))^n$$

$$= \sum_{n=0}^{N} |a_n||x|^n + \lim_{l\to\infty} \left((\frac{1}{2}(M|x|+1))^{N+1} \left(\frac{1-(\frac{1}{2}(M|x|+1))^{l-N}}{1-(\frac{1}{2}(M|x|+1))} \right) \right)$$

$$= \sum_{n=0}^{N} |a_n||x|^n + \left((\frac{1}{2}(M|x|+1))^{N+1} \left(\frac{1}{1-(\frac{1}{2}(M|x|+1))} \right) \right)$$

Hence by comparison test $\sum_{n=0}^{\infty} |a_n x^n|$ converges.

3. Similar to part 2 above, we can find $N \in \mathbb{N}$ such that for all $n > N$, $|a_n|^{1/n} < \frac{1}{2}(M+1/r)$. For any $\epsilon > 0$, let $N' \in \mathbb{N}$ satisfies $(\frac{1}{2}(rM+1))^{N'} < \frac{1}{2}(1-rM)\epsilon$, and let $N_\epsilon = \max(N, N')$. Then for any $m > n > N_\epsilon$, then

$$\sup\{|\sum_{i=0}^{m} a_i x^i - \sum_{i=0}^{n} a_i x^i| : |x| \leq r\} \leq \sum_{i=n+1}^{m} |a_i| r^i$$

$$\leq \sum_{i=n+1}^{m} (Mr/2+1/2)^i = (Mr/2+1/2)^{n+1} \cdot \frac{1-(Mr/2+1/2)^{m-n}}{1/2-Mr/2}$$

$$< \frac{(Mr/2+1/2)^{n+1}}{1/2-rM/2} < \epsilon$$

□

Example 5.2.10. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ converges uniformly on any finite interval. This is because

$$a_n = \begin{cases} 0 & n \text{ even} \\ \frac{1}{n!} & n \text{ odd} \end{cases}$$

So for any natural number k, any $n > k$,

$$|a_n| \leq \frac{1}{n!} \leq \frac{1}{k^{n-k}}$$

$$\limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} \frac{1}{k^{(n-k)/n}} = \frac{1}{k}$$

Because k can be arbitrarily large, $\limsup_{n\to\infty} |a_n|^{1/n} = 0$

Example 5.2.11. $\sum_{n=0}^{\infty} n^2 x^{n^2}$ converges when $|x| < 1$.

5.2.3 Summation by Parts

A trick we often use to show convergence of series is the following:

Lemma 5.2.12. Let $\{a_n\}$, $\{b_n\}$ be sequences of real numbers, $0 \le m < n$, $s_j = \sum_{i=0}^{j} a_i$. Then $\sum_{i=m}^{n} a_i b_i = s_n b_n - s_{m-1} b_m + \sum_{i=m}^{n-1} s_i (b_i - b_{i+1})$.

Proof.

$$\sum_{i=m}^{n} a_i b_i = \sum_{i=m}^{n} (s_i - s_{i-1}) b_i = s_n b_n - s_{m-1} b_m + \sum_{i=m}^{n-1} s_i (b_i - b_{i+1})$$

□

As a consequence, we have:

Theorem 5.2.13. Let $\{a_n\}$, $\{b_n\}$ be sequences of real numbers, let $s_j = \sum_{i=0}^{j} a_i$. If there is some M such that $|s_j| \le M$ for all j, and if $b_{n+1} \le b_n$ for all n, $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=0}^{\infty} a_n b_n$ converges.

Proof. By Theorem 5.2.3, we only need to show that for any ϵ, there is some N such that for any $N < m \le n$, $|\sum_{i=m}^{n} a_i b_i| < \epsilon$. Because $\{b_n\}$ is non-decreasing and $\lim_{n\to\infty} b_n = 0$, $b_n \ge 0$ for all n, and there is some N such that when $n > N$, $0 \le b_n < \frac{\epsilon}{10M}$. Because $a_n = s_n - s_{n-1}$, and all s_j has absolute value no more than M, $|a_n| \le 2M$ for all n.

If $N < m = n$,

$$|\sum_{i=m}^{n} a_i b_i| = |a_m||b_m| \le 2M \cdot \frac{\epsilon}{10M} < \epsilon$$

If $N < m < n$, by Lemma 5.2.12,

$$|\sum_{i=m}^{n} a_i b_i| = |s_n b_n - s_{m-1} b_m + \sum_{i=m}^{n-1} s_i (b_i - b_{i+1})|$$

$$\le |s_n||b_n| + |s_{m-1}||b_m| + \sum_{i=m}^{n-1} |s_i| (b_i - b_{i+1})$$

$$\le M \cdot \frac{\epsilon}{10M} + M \cdot \frac{\epsilon}{10M} + M \sum_{i=m}^{n-1} (b_i - b_{i+1})$$

$$= \frac{\epsilon}{5} + M(b_m - b_n) \le \frac{\epsilon}{5} + M b_m \le \frac{\epsilon}{5} + \frac{\epsilon}{10} < \epsilon$$

□

Example 5.2.14. When $a_n = (-1)^n$, $b_n \geq 0$, Theorem 5.2.13 gives the convergence criteria for alternating series.

Example 5.2.15.

$$\left|\sum_{i=0}^{n} \sin(i)\right| = \left|\sum_{i=0}^{n} \frac{\cos(i-1/2)-\cos(i+1/2)}{2\sin(1/2)}\right|$$

$$= \left|\frac{\cos(-1/2)-\cos(n+1/2)}{2\sin(1/2)}\right| \leq \frac{1}{\sin(1/2)}$$

Now Theorem 5.2.13 shows that $\sum_{n=0}^{\infty} \frac{\sin(n)}{\log(n+2)}$ converges.

5.3 Discrete L'Hospital's Rule

Theorem 5.3.1. Let $\{a_n\}$, $\{b_n\}$ be two sequences of real numbers, $b_{n+1} > b_n > 0$ for all n, and $\{b_n\}$ has no upper bound. Then if $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = L$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

Proof. For any $\epsilon > 0$, let N_1 be such that when $n \geq N_1$, $|\frac{a_{n+1}-a_n}{b_{n+1}-b_n} - L| < \epsilon/3$, let N be such that $N > N_1$ and $b_N > \frac{3|a_{N_1}-Lb_{N_1}|}{\epsilon}$. Then for any $n > N$,

$$\left|\frac{a_n}{b_n} - L\right| = \left|\frac{(a_{N_1} - Lb_{N_1}) + \sum_{i=N_1}^{n-1}((a_{i+1}-a_i) - L(b_{i+1}-b_i))}{b_n}\right|$$

$$\left|\frac{a_{N_1} - Lb_{N_1}}{b_n}\right| + \sum_{i=N_1}^{n-1} \left|\frac{(a_{i+1}-a_i) - L(b_{i+1}-b_i)}{b_n}\right|$$

$$< \frac{\epsilon}{3} \cdot \frac{b_N}{b_n} + \frac{\epsilon}{3} \cdot \frac{b_n - b_{N_1}}{b_n} < \epsilon$$

□

Example 5.3.2. If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} \frac{\sum_{i=0}^{n} a_i}{n+1} = L$.

6 Differentiation and integration

6.1 Definition of differentiation

Definition 6.1.1. Let f be a real-valued function defined on an open set $U \subseteq \mathbb{R}$. We say that the **derivative** of f at $x \in U$ is L, denoted as $f'(x) = L$ or $\frac{d}{dx}f|_x = L$, iff the function

$$g(h) = \begin{cases} L & h = 0 \\ \frac{1}{h}(f(x+h) - f(x)) & h \neq 0 \end{cases}$$

is continuous at 0.

Remark 6.1.2. By Definition 4.2.9, we can also state this as $\lim_{h\to 0} \frac{1}{h}(f(x+h) - f(x)) = L$. If f has derivative on every point in a set $A \subseteq \mathbb{R}$ we say f is differentiable on A.

Theorem 6.1.3. (Linear Approximation) $f'(x) = L$ iff for any $\epsilon > 0$, there is some $r > 0$, such that when $t \in (x - r, x + r)$,

$$|f(t) - f(x) - L(t - x)| \leq \epsilon |t - x|$$

Proof. By Definition 6.1.1, $f'(x) = L$ iff for any $\epsilon > 0$, there is some $r > 0$ such that $0 < |h| < r$ implies $|\frac{1}{h}(f(x+h) - f(x)) - L| \leq \epsilon$. Replace $x + h$ with t and multiply both sides by $|t - x|$ one can see that it is identical to the inequality in the Theorem. □

Remark 6.1.4. An immediate consequence is that if f is differentiable at x then it is also continuous at x.

Remark 6.1.5. The condition in Theorem 6.1.3 can also be written as $f(t) = f(x) + L(t - x) + o(|t - x|)$.

One can use Theorem 6.1.3 to prove the various properties of derivatives that we have seen in Calculus. For example the chain rule:

Theorem 6.1.6. Let f, g be two real valued functions on $\mathbb{R}$, $f'(x) = L$, $g'(f(x)) = L'$, then $(g \circ f)'(x) = L'L$.

Proof. Let $r_1(t) = f(t) - f(x) - L(t - x)$, $r_2(t) = g(t) - g(f(x)) - L'(t - f(x))$, $r(t) = g(f(t)) - g(f(x)) - L'L(t - x)$. Then

$$r(t) = g(f(x)) + L'(f(t) - f(x)) + r_2(f(t)) - g(f(x)) - L'L(t - x)$$

$$= g(f(x)) + L'(L(t-x) + r_1(t)) + r_2(f(x) + L(t-x) + r_1(t)) - g(f(x)) - L'L(t-x)$$

$$= L'r_1(t) + r_2(f(x) + L(t - x) + r_1(t))$$

For any $\epsilon > 0$, pick $\delta > 0$ such that when $t \in (x - \delta, x + \delta)$,

$$r_1(t) \leq \frac{\epsilon}{3|L'|}|t - x|$$

Find $\delta' > 0$ such that when $t \in (f(x) - \delta', f(x) + \delta')$,

$$r_2(t) \leq \frac{\epsilon}{3(|L| + \frac{\epsilon}{3|L'|})}|t - f(x)|$$

Find $\delta'' > 0$ such that when $|t - x| < \delta''$ then $|f(t) - f(x)| < \delta'$. Then for any $t \in (x - \min(\delta, \delta''), x + \min(\delta, \delta''))$, we have

$$r(t) \leq \epsilon/3 + \epsilon/3 \leq \epsilon$$

□

6.2 Mean Value Theorem

Theorem 6.2.1. Let f be a real valued function defined on some open set $U \subseteq \mathbb{R}$, $x \in U$.

1. If $f'(x) > 0$, there is $r > 0$ such that for any $x' \in (x, x + r)$, $f(x') > f(x)$, for any $x' \in (x - r, x)$, $f(x') < f(x)$.

2. If $f'(x) < 0$, there is $r > 0$ such that for any $x' \in (x, x + r)$, $f(x') < f(x)$, for any $x' \in (x - r, x)$, $f(x') > f(x)$.

3. If f takes maximum or minimum at x (i.e. $f(x) = \sup f(I)$ or $f(x) = \inf f(I)$), then $f'(x) = 0$.

Proof. Suppose $f'(x) > 0$, by Theorem 6.1.3, there is $r > 0$ such that $t \in (x - r, x + r)$ implies

$$|f(t) - f(x) - f'(x)(t - x)| \leq \frac{f'(x)}{2}(t - x)$$

Hence $f(t)$ lies in between $f'(x) + \frac{f'(x)}{2}(t - x)$ and $f'(x) + \frac{3f'(x)}{2}(t - x)$, which implies that when $x + r > t > x$, $f(t) > f(x)$, and when $x - r < t < x$, $f(t) < f(x)$. The case when $f'(x) < 0$ is analogous. Part 3 of the theorem follows from parts 1 and 2. □

Theorem 6.2.2. (Intermediate Value Theorem for derivatives) If f is differentiable on an open interval I, $a < b$ two elements of I, such that $f'(a) < m < f'(b)$, then there is some $c \in (a, b)$ where $f'(c) = m$.

Proof. Consider $g(x) = f(x) - mx$. g is continuous on compact set $[a, b]$ hence takes minimum somewhere on $[a, b]$. It can not take minimum at a or b because of Theorem 6.2.1 $g'(a) < 0$ and $g'(b) > 0$, hence the minimum has to be taken at some $c \in (a, b)$, hence by Theorem 6.2.1, $0 = g'(c) = f'(c) - m$. □

Remark 6.2.3. The Theorem is also true if $f'(a) > m > f'(b)$, with an almost identical proof.

Remark 6.2.4. The derivative of a function may not be continuous. For example

$$f(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin(1/x^3) & x \neq 0 \end{cases}$$

The following theorems are usually called **"Mean Value Theorem"**:

Theorem 6.2.5. If f is a real-valued continuous function that is continuous on $[a, b]$, differentiable on (a, b), $f(a) = f(b)$, then there is $c \in (a, b)$ such that $f'(c) = 0$.

Proof. The function f is continuous on compact set $[a, b]$, hence by Theorem 3.4.9, there are $x_M, x_m \in [a, b]$ where $f(x_m) = \inf(f([a, b]))$ and $f(x_M) = \sup(f([a, b]))$.

Suppose both x_m and x_M are in $\{a, b\}$, then f is constant and $f'(c) = 0$ for all $c \in (a, b)$. If either x_m or x_M is in (a, b), by Theorem 6.2.1, f' is 0 at that point. □

Theorem 6.2.6. Let f be a real valued function, continuous on $[a, b]$ and differentiable on (a, b). Then there is some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Apply Theorem 6.2.5 to $F(x) = f(x) - \frac{f(b)-f(a)}{b-a}x$ □

Theorem 6.2.7. Let f, g be two real valued functions that are continuous on $[a, b]$ and differentiable on (a, b). Then there is some $c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

Proof. Apply Theorem 6.2.5 to the function

$$F(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$$

□

Remark 6.2.8. As a consequence, if on some open interval I we have $g' > 0$ and $ag'(x) \leq f'(x) \leq bg'(x)$, then for any $c \in I$, any $x \in I$ we have

$$|f(x) - f(c) - \frac{a + b}{2}(g(x) - g(c))| \leq \frac{b - a}{2}|g(x) - g(c)|$$

6.2.1 L'Hospital's Rule

Definition 6.2.9.

- Let f be a real-valued function defined on (c, ∞). We say that $\lim_{x\to\infty} f(x) = L$ iff for any $\epsilon > 0$, there is some $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$.

- Let f be a real valued function defined on $(0, c)$. We say that $\lim_{x\to 0^+} f(x) = L$ if for any $\epsilon > 0$, there is $\delta > 0$, such that $0 < x < \delta$ implies $|f(x)-L| < \epsilon$.

Theorem 6.2.10. (L'Hospital's Rules)

1. Let f, g be differentiable functions on (c, ∞). Suppose $g > 0$, $g' > 0$ and g is unbounded, and also $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.
2. Let f, g be differentiable functions on (c, ∞). Suppose $g > 0$, $g' < 0$, $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f(x) = 0$, and $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.
3. Let f, g be differentiable functions on $(0, c)$. Suppose $g > 0$, $g' < 0$ and g is unbounded, and also $\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to 0^+} \frac{f(x)}{g(x)} = L$.
4. Let f, g be differentiable functions on $(0, c)$. Suppose $g > 0$, $g' > 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} g(x) = 0$, and also $\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to 0^+} \frac{f(x)}{g(x)} = L$.

Proof. The proof for all 4 states are similar and also all similar to the proof of Theorem 5.3.1. We will only do 2 as an example.

For any $\epsilon > 0$, find M such that when $x > M$, $|\frac{f'(x)}{g'(x)} - L| < \epsilon/10$. Now for any $x > M$, let $M' > x$ be large enough such that $\frac{|f(M')-Lg(M')|}{g(x)} < \frac{\epsilon}{10}$. Then

$$|\frac{f(x)}{g(x)} - L| = \frac{|f(x) - f(M') - L(g(x) - g(M'))| + |f(M') - Lg(M')|}{g(x)} < \frac{\epsilon}{5}$$

The last $<$ is due to Remark 6.2.8. □

6.3 Definition of integration

Definition 6.3.1. Let $I = [a, b]$ be a finite closed interval. For any finite subset S of (a, b), the **partition** induced by S is as follows: suppose S has $n-1$ elements, ordered from smallest to largest as $x_1, \dots, x_{n-1}$, then $P(S) = (x_0 = a, x_1, \dots, x_{n-1}, x_n = b)$.

Definition 6.3.2. Let f be a bounded real valued function on $[a, b]$, the **upper sum** of f with respect to partition $P = (x_0, \dots, x_n)$ is

$$U(f, P) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \sup(f([x_i, x_{i+1}]))$$

The **lower sum** of f with respect to partition $P = \{x_0, \dots, x_n\}$ is

$$L(f, P) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \inf(f([x_i, x_{i+1}]))$$

Lemma 6.3.3. If $S \subseteq S' \subseteq (a, b)$ are two finite sets, then $U(f, P(S')) \leq U(f, P(S))$.

Proof. By induction on the size of $S' \backslash S$, we only need to show the inequality for when S' has one more element than S. Suppose $P(S) = (x_0, \ldots, x_n)$, and this extra element x' is between x_j and x_{j+1}, then

$$U(f, P(S')) = \sum_{i=0}^{j-1}(x_{i+1}-x_i)\sup(f([x_i, x_{i+1}]))+\sum_{i=j+1}^{n-1}(x_{i+1}-x_i)\sup(f([x_i, x_{i+1}]))$$

$$+(x' - x_j)\sup(f([x_j, x'])) + (x_{j+1} - x')\sup(f([x', x_{j+1}]))$$

Because

$$\sup(f([x_j, x'])) \leq \sup(f([x_j, x_{j+1}]))$$
$$\sup(f([x', x_{j+1}])) \leq \sup(f([x_j, x_{j+1}]))$$

The sum of the last two terms in the summation formula above is no more than $(x_{j+1} - x_j)\sup(f([x_j, x_{j+1}]))$, hence

$$U(f, P(S')) \leq U(f, P(S))$$

□

Remark 6.3.4. The same argument can be used to show that if $S \subseteq S' \subseteq (a, b)$ be two finite sets, then $L(f, P(S')) \geq L(f, P(S))$.

Theorem 6.3.5. Suppose P and P' are two partitions of $I = [a, b]$, then $L(f, P) \leq U(f, P')$.

Proof. Suppose $P = P(S)$, $P' = P(S')$, then Lemma 6.3.3 and Remark 6.3.4 implies that

$$L(f, P) = L(f, P(S)) \leq L(f, P(S \cup S')) \leq U(f, P(S \cup S'))$$
$$\leq U(f, P(S')) = U(f, P')$$

□

Remark 6.3.6. Theorem 6.3.5 implies that

$$\sup\{L(f, P) : P \text{ partition of } I\} \leq \inf\{U(f, P) : P \text{ partition of } I\}$$

Definition 6.3.7. Let f be a bounded real-valued function on finite closed interval $I = [a, b]$. If

$$\sup\{L(f, P) : P \text{ partition of } I\} = \inf\{U(f, P) : P \text{ partition of } I\} = C$$

We call f **Riemann integrable** on I, and define C as the **Riemann integral** of f on I, denoted as

$$C = \int_I f(x)dx = \int_a^b f(x)dx$$

It is easy to see that

Theorem 6.3.8. f is Riemann integrable on $I = [a, b]$, iff for any $\epsilon > 0$, there is a partition $P = (x_0, \dots, x_n)$ of I, such that

$$U(f, P) - L(f, P) = \sum_{i=0}^{n-1} (x_{i+1} - x_i)(\sup([x_i, x_{i+1}]) - \inf([x_i, x_{i+1}])) < \epsilon$$

And

$$L(f, P) \leq \int_a^b f(x)dx \leq U(P, f)$$

□

Example 6.3.9. The function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable, because $U(f, P) = 1$, $L(f, P) = 0$ for any partition P.

6.3.1 Integrability of continuous functions

Definition 6.3.10. Let (X, d), (Y, d') be two metric spaces. We say a function $f : X \to Y$ is **uniformly continuous** if for any $\epsilon > 0$, there is $\delta > 0$, such that for any $p, q \in X$, if $d(p, q) < \delta$ then $d'(f(p), f(q)) < \epsilon$.

Example 6.3.11. Any Lipschitz function is uniformly continuous. $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 1/x$ is continuous but not uniformly continuous.

Theorem 6.3.12. If $f : [a, b] \to \mathbb{R}$ is uniformly continuous then it is Riemann integrable.

Proof. For any $\epsilon > 0$, pick natural number N sufficiently large such that when $|x - y| < \frac{b-a}{N}$, $|f(x) - f(y)| < \frac{\epsilon}{3(b-a)}$. Now apply Theorem 6.3.8 using the partition

$$P = \left(a, a + \frac{b-a}{N+1}, a + \frac{2(b-a)}{N+1}, \dots, b\right)$$

we get $U(f, P) - L(f, P) < \epsilon$. □

Theorem 6.3.13. Let (X, d) and (Y, d') be two metric spaces, if X is compact and $f : X \to Y$ is continuous then it is uniformly continuous.

Proof 1. For any $\epsilon > 0$, consider the subset $V_\epsilon \subseteq X \times X$ defined as

$$V_\epsilon = \{(a, b \in X \times X : d'(f(a), f(b)) \geq \epsilon\}$$

We shall show that V_ϵ is closed in $(X \times X, d_{sup})$: for any $(p,q) \notin V_\epsilon$, by construction $d'(f(p), f(q)) < \epsilon$. Let

$$\epsilon' = \frac{\epsilon - d'(f(p), f(q))}{3}$$

Let $\delta_1 > 0$ satisfies $d(x,p) < \delta_1$ implies $d'(f(x), f(p)) < \epsilon'$, $\delta_2 > 0$ satisfies $d(x,q) < \delta_2$ implies $d'(f(x), f(q)) < \epsilon'$. Then, as long as $d(x,p) < \min(\delta_1, \delta_2)$, $d(x',p) < \min(\delta_1, \delta_2)$,

$$\begin{aligned} d'(f(x), f(x')) &\leq d'(f(x), f(p)) + d'(f(p), f(q)) + d'(f(q), f(x')) \\ &< d'(f(p), f(q)) + 2\epsilon' < \epsilon \end{aligned}$$

In other words, $B_{(X\times X, d_{sup})}((p,q), \min(\delta_1, \delta_2)) \cap V_\epsilon = \emptyset$.

By Theorem 3.4.11, $(X \times X, d_{sup})$ is compact, hence by Theorem 3.4.10, so is V_ϵ. The function $(x,y) \mapsto d(x,y)$ is a positive continuous function on V_ϵ (because it is 2-Lipschitz), hence by Theorem 3.4.9 it has a positive minimum $\delta = d(x_m, y_m)$, where $(x_m, y_m) \in V_\epsilon$. In other words, if $d(x,y) < \delta$ then $(x,y) \notin V_\epsilon$, hence $d'(f(x), f(y)) < \epsilon$. This shows the uniform continuity of f. □

Proof 2. For any $\epsilon > 0$, any $x \in X$, let $r_x > 0$ satisfies $d(x',x) < r_x$ implies $d'(f(x'), f(x)) < \epsilon/2$. Then $X = \bigcup_{x\in X} B_X(x, r_x/2)$. By compactness, there are finitely many $x_1, \dots, x_n \in X$ such that $\bigcup_{i=1}^n B_x(x_i, r_{x_i}/2) = X$. Let $\delta = \min\{r_{x_1}, \dots, r_{x_n}\} > 0$. For any $x, x' \in X$, if $d(x,x') < \delta$, let $x_i \in \{x_1, \dots, x_n\}$ such that $x \in B_X(x_i, r_{x_i}/2)$, then

$$d(x, x_i) < r_{x_i}/2 < r_{x_i}$$

$$d(x', x_i) < d(x, x_i) + d(x', x) < r_{x_i}/2 + \delta \leq r_{x_i}$$

Hence

$$d'(f(x), f(x_i)) < \epsilon/2$$
$$d'(f(x'), f(x_i)) < \epsilon/2$$

By triangle inequality $d'(f(x), f(x')) < \epsilon$. □

Theorem 6.3.14. If $f : [a,b] \to \mathbb{R}$ is continuous then it is Riemann integrable.

Proof. This follows from Theorem 6.3.13 and Theorem 6.3.12. □

Remark 6.3.15. (Please don't use this fact in HW and exam) One can also show that a bounded function on a finite closed interval $[a,b]$ is Riemann integrable iff it is "almost everywhere continuous", i.e. for any $\epsilon > 0$, there are countably infinitely many intervals $I_1, I_2, \dots$, such that $\sum_{i=1}^\infty |I_i| < \epsilon$ and f is continuous at $[a,b]\backslash \bigcup_{i=1}^\infty I_i$. For example, the function defined on $[0,1]$:

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ |q|^{-1/2} & x = p/q, \gcd(p,q) = 1 \end{cases}$$

is Riemann integrable.

6.4 Fundamental Theorem of Calculus

Theorem 6.4.1. If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, $c \in (a, b)$, then $f|_{[a,c]}$ and $f|_{[c,b]}$ are both Riemann integrable on $[a, c]$ and $[c, b]$ respectively, and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Proof. By Theorem 6.3.8, for any $\epsilon > 0$ there is a partition $P = P(S)$ such that

$$U(f, P) - L(f, P) < \epsilon$$

By Lemma 6.3.3 and Remark 6.3.4, if $P' = P(S \cup \{c\})$ then

$$U(f, P') - L(f, P') \leq U(f, P) - L(f, P) < \epsilon$$

In particular, if $P' = (x_0 = a, \ldots, x_k = c, \ldots, x_n = b)$, let $P'_1 = (x_0 = a, \cdots x_k = c)$ and $P'_2 = (x_k = c, \ldots, x_n = b)$, then P'_1 and P'_2 are partitions of $[a, c]$ and $[c, b]$ respectively, and

$$U(f, P') - L(f, P') = (U(f, P'_1) - L(f, P'_1)) - (U(f, P'_2) - L(f, P'_2))$$

Hence

$$U(f, P'_1) - L(f, P'_1) < \epsilon$$
$$U(f, P'_2) - L(f, P'_2) < \epsilon$$

These implies that $f|_{[a,c]}$ and $f|_{[c,b]}$ are Riemann integrable on $[a, c]$ and $[c, b]$ respectively.

Furthermore, because

$$L(f, P') \leq \int_a^b f(x)dx \leq U(f, P')$$

$$L(f, P'_1) \leq \int_a^c f(x)dx \leq U(f, P'_1)$$

$$L(f, P'_2) \leq \int_c^b f(x)dx \leq U(f, P'_2)$$

We have

$$|\int_a^b f(x)dx - \int_a^c f(x)dx - \int_c^b f(x)dx| \leq \epsilon$$

Because ϵ can be any positive real number,

$$\int_a^b f(x)dx - \int_a^c f(x)dx - \int_c^b f(x)dx = 0$$

□

Furthermore, from the definition of Riemann integrals we can show:

Theorem 6.4.2. If $f, g : [a, b] \to \mathbb{R}$ are two Riemann integrable functions. If for any $x \in [a, b]$, $f(x) \le g(x)$, then

$$\int_a^b f(x)dx \le \int_a^b g(x)dx$$

Proof. For any $\epsilon > 0$, pick partition P such that $U(f, P) - L(f, P) < \epsilon$. By definition, $L(f, P) \le L(g, P)$, hence

$$\int_a^b f(x)dx - \int_a^b g(x)dx \le U(f, P) - L(g, P)$$
$$\le (U(f, P) - L(f, P)) + (L(f, P) - L(g, P)) \le \epsilon$$

Because ϵ can be any positive number,

$$\int_a^b f(x)dx \le \int_a^b g(x)dx$$

□

Remark 6.4.3. Actually if $f(x) < g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)dx < \int_a^b g(x)dx$, but the proof is far more complicated.

Theorem 6.4.4. If f is a real valued continuous function on $[a, b]$, then

$$F : x \mapsto \int_a^x f(t)dt$$

is continuous on $[a, b]$, differentiable on (a, b), and $F'(x) = f(x)$ for any $x \in (a, b)$.

Proof. f is continuous on compact set hence bounded. Let M be an upper bound of $|f|$. For any $x, x' \in [a, b]$, if $x < x'$ then

$$|F(x') - F(x)| = |\int_x^{x'} f(t)dt| \le M|x' - x|$$

in other words F is M-Lipschitz hence continuous.

For any $c \in (a, b)$, any $\epsilon > 0$, let $r > 0$ satisfies $f((c - r, c + r)) \subseteq (f(c) - \epsilon/2, f(c) + \epsilon/2)$, hence for any $0 < h < r$,

$$F(c + h) - F(c) = \int_c^{c+h} f(t)dt \le \int_c^{c+h} (f(c) + \epsilon)dt = hf(c) + \epsilon h/2$$

Similarly, $F(c + h) - F(c) \ge f(c)h - \epsilon h/2$. Hence

$$\left| \frac{F(c + h) - F(c)}{h} - f(c) \right| \le \epsilon/2 < \epsilon$$

We can similarly prove that the inequality above is true when $-r < h < 0$, hence F is differentiable at c and $F'(c) = f(c)$. □

Theorem 6.4.5. If F is continuous on $[a,b]$, differentiable on (a,b), f is a Riemann integrable function on $[a,b]$ and $F'(x) = f(x)$ for any $x \in (a,b)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof. For any $\epsilon > 0$, let $P = (x_0 = a, x_1, \ldots, x_n = b)$ be a partition of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$, then by Mean Value Theorem,

$$F(b)-F(a) = \sum_{i=1}^n (F(x_i)-F(x_{i-1})) \leq \sum_{i=1}^n (x_n - x_{n-1}) \sup(F'([x_{n-1}, x_n])) = U(f,P)$$

Similarly $F(b) - F(a) \geq L(f,P)$, hence $|\int_a^b f(x)dx - (F(b) - F(a))| < \epsilon$. This proved the theorem. □

Remark 6.4.6. $F_1(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin(1/x) & x \neq 0 \end{cases}$ is differentiable on $[-1,1]$, the derivative is Riemann integrable but not continuous. $F_2(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin(1/x^2) & x \neq 0 \end{cases}$ is differentiable on $[-1,1]$ and the derivative is unbounded hence not Riemann integrable on $[-1,1]$.

6.5 Relationship with uniform convergence

Theorem 6.5.1. Let $\{f_n\}$ be a sequence of real valued functions on $I = [a,b]$ that converges uniformly to g, f_n are all Riemann integrable, then so is g.

Proof. g is bounded because all f_n are bounded. For any $\epsilon > 0$, find N such that $d_{sup}(f_N, g) < \frac{\epsilon}{4(b-a)}$, find partition P of $I = [a,b]$ such that $U(f_N, P) - L(f_N, P) < \epsilon/2$, then

$$U(g,P) - L(g,P) \leq |U(g,P) - U(f_N,P)| + |L(g,P) - L(f_N,P)|$$

$$+|U(f_N,P) - L(f_N,P)| < \epsilon$$

Hence by Theorem 6.3.8, g is Riemann integrable. □

Theorem 6.5.2. Let R be the set of Riemann integrable functions in $BF([a,b],\mathbb{R})$ (by Theorem 6.5.1 R is closed under d_{sup}). The function $I : R \to \mathbb{R}$ defined as $I(f) = \int_a^b f(x)dx$ is $(b-a)$-Lipschitz under d_{sup} on R and Euclidean metric on $\mathbb{R}$.

Proof. If $d_{sup}(f,g) = r$, then $f(x) - r \leq g(x) \leq f(x) + r$, hence this follows from Theorem 6.4.2 and the fact that $\int_a^b (f(x) + c)dx = \int_a^b f(x)dx + c(b-a)$. □

An immediate consequence of Theorem 6.5.1 and Theorem 6.5.2 is the following:

Theorem 6.5.3. If $\{f_n\}$ is a sequence of Riemann integrable functions on $[a, b]$, f_n converges to g uniformly, then g is also Riemann integrable and $\int_a^b g(x)dx = \lim_{n\to\infty} \int_a^b f(x)dx$.

□

Example 6.5.4. Let $I = [0, 1]$, consider functions g, f_n on I defined as

$$g(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q}, \gcd(p, q) = 1, q > 0 \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$$f_n(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q}, \gcd(p, q) = 1, 0 < q < n \\ 0 & \text{otherwise} \end{cases}$$

Then f_n are all Riemann integrable (because they are continuous except for finitely many points), $\int_0^1 f_n = 0$, and f_n converges to g uniformly. Hence g is Riemann integrable and $\int_0^1 g(x)dx = 0$.

Example 6.5.5. The functions

$$f_n = \begin{cases} nx^n & x < 1 \\ 0 & x = 1 \end{cases}$$

defined on $[0, 1]$ converges to the 0 function pointwise as $n \to \infty$, but the convergence is not uniform. And

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1 \neq \int_0^1 0dx$$

Example 6.5.6. Let R be the set of Riemann integrable functions in $BF([a, b], \mathbb{R})$, the function $J : R \to R$ defined as

$$J(f) = \int_a^x f(t)dt$$

is also $(b - a)$-Lipschitz under d_{sup}. Hence if f_n converges to g uniformly over $[a, b]$, and f_n are all Riemann integrable, then $\{\int_a^x f_n(t)dt\}$ converges to $\int_a^x g(t)dt$ uniformly.

Theorem 6.5.7. Let $I = [a, b]$ be a finite closed interval, $c \in I$, f_n and g continuous on I and differentiable on its interior. Suppose $\lim_{n\to\infty} f_n(c) = g(c)$, and f_n' converges to g' uniformly on the interior of I, then f_n converges to g uniformly on I.

Proof. For any $\epsilon > 0$, let N satisfies $n > N$ implies $|f_n(c) - g(c)| < \epsilon/2$, N' satisfies $n > N'$ implies $|f_n'(x) - g'(x)| < \frac{\epsilon}{2(b-a)}$ for any $x \in (a, b)$, then for any $n > \max(N, N')$, by Theorem 6.2.6, for any $x \neq c$,

$$(f_n - g)(x) - (f_n - g)(c) = (x - c)(f_n' - g_n')(d)$$

where d lies in between x and c, hence is in (a,b). Hence

$$|f_n(x)-g(x)| \le |f_n(c)-g(c)| + |x-c||f_n'(d)-g_n'(d)| < \epsilon$$

□

6.5.1 Application to power series

Consider power series $f(x)=\sum_{n=0}^\infty a_n x^n$. Theorem 5.2.9 implies that if $M = \limsup_{n\to\infty}|a_n|^{1/n} < \infty$, then for any $r>0$, if $Mr<1$, then the partial sums $p_n(x)=\sum_{i=0}^n a_i x^i$ converges uniformly on $[-r,r]$ to $f(x)$. Apply Example 6.5.6 to $\{p_n\}$ and f, we get:

Theorem 6.5.8. Let $f(x)=\sum_{n=0}^\infty a_n x^n$, $r>0$ and $r\limsup_{n\to\infty}|a_n|^{1/n}<1$, then for any $x\in[0,r]$,

$$\int_0^x f(t)dt = \sum_{n=0}^\infty \frac{a_n}{n+1}x^{n+1}$$

For any $x\in[-r,0]$,

$$\int_x^0 f(t)dt = -\sum_{n=0}^\infty \frac{a_n}{n+1}x^{n+1}$$

□

Remark 6.5.9. When $a>b$, we often denote $-\int_b^a f(x)dx$ as $\int_a^b f(x)dx$. Hence the above theorem can be stated as

$$\int_0^x f(t)dt = \sum_{n=0}^\infty \frac{a_n}{n+1}x^{n+1}$$

Furthermore, one can verify using definition of lim sup that

$$\limsup_{n\to\infty}|a_n|^{1/n} = M$$

iff for any $\epsilon>0$, there is some N such that for any $n>N$, $|a_n|^{1/n}<M+\epsilon$, and there is some $m>n$ such that $|a_m|^{1/m}>M-\epsilon$. Because $\lim_{n\to\infty}(n+1)^{1/n}=1$,

$$\limsup_{n\to\infty}|(n+1)a_{n+1}|^{1/n} = M$$

Hence $p_n'(x)$ also converges uniformly on $[-r,r]$ to $\sum_{n=0}^\infty (n+1)a_{n+1}x^n$.

Because $\sum_{n=0}^\infty (n+1)a_{n+1}x^n$ is continuous, by Theorem 6.4.4 there is some function g continuous on $[-r,r]$ such that $g'(x)=\sum_{n=0}^\infty (n+1)a_{n+1}x^n$ for $x\in(-r,r)$ and $g(0)=a_0$. By Theorem 6.5.7, p_n converges to g uniformly on $[-r,r]$, hence $g=\sum_{n=0}^\infty a_n x^n = f$. This shows that

Theorem 6.5.10. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $r > 0$ and $r \limsup_{n\to\infty} |a_n|^{1/n} < 1$, then for any $x \in (-r, r)$

$$f'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

□

Example 6.5.11. When $|x| < 1$,

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

By Theorem 6.5.8, for any $0 < r < 1$,

$$\log(1+r) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} r^n$$

Given any $\epsilon > 0$, if $\frac{1}{N} < \epsilon/3$, then by alternating series test (from what you might have seen in Calc 2, or you can also deduce this by using summation by parts)

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} - \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} \right| < \epsilon/3$$

and for all $0 < r < 1$,

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^{n-1} r^n}{n} - \sum_{n=1}^{N} \frac{(-1)^{n-1} r^n}{n} \right| < \epsilon/3$$

Furthermore, there is some $\delta > 0$ such that when $r \in (1-\delta, 1)$,

$$\left| \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} - \sum_{n=1}^{N} \frac{(-1)^{n-1} r^n}{n} \right| < \epsilon/3$$

This shows that for all $r \in (1-\delta, 1)$,

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} - \log(1+r) \right| < \epsilon$$

This is only possible when

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log(2)$$

6.6 Taylor's Theorem

Definition 6.6.1. The n-th order derivative of a function f is defined as

$$f^{(0)} = f$$
$$f^{(n+1)} = (f^{(n)})'$$

Let f be a function with n-th order derivatives on $(-r, r)$, then

$$f(x) - \sum_{i=0}^{n} \frac{f^{(n)}(0)}{n!} x^n$$

has n-th order derivatives on $(-r, r)$, and all of its higher order derivatives up to order n are 0 at 0. This is called **the n-th Taylor remainder** of f, denoted as R_n, and Taylor's Theorems are different ways of estimating this R_n.

Theorem 6.6.2. If f has $n+1$-th order derivative on $(0, x)$, then there is $c \in (0, x)$ such that

$$R_n(x) = \frac{f^{(n+1)}(c)x^{n+1}}{(n+1)!}$$

When $n = 0$ one can see that this just reduces to Theorem 6.2.6.

Proof. Apply Theorem 6.2.7 to $f(x) = R_n(x)$ and $g(x) = x^{n+1}$, then we get c_1 between 0 and x such that

$$(n+1)c_1^n(R_n(x) - 0) = R_n'(c_1)(x^{n+1} - 0)$$

So

$$\frac{R_n(x)}{x^{n+1}} = (n+1)\frac{R_n'(c_1)}{c_1^n}$$

Apply Theorem 6.2.7 to $f(x) = R_n'(x)$ and $g(x) = x^n$, we get c_2 between 0 and c_1 such that

$$\frac{R_n'(c_1)}{c_1^n} = n\frac{R_n''(c_2)}{c_2^{n-1}}$$

Repeat this process, we get

$$\frac{R_n(x)}{x^{n+1}} = (n+1)\frac{R_n'(c_1)}{c_1^n} = (n+1)n\frac{R_n''(c_2)}{c_2^{n-1}} = \dots$$
$$= (n+1)!R^{(n+1)}(c_{n+1}) = (n+1)!f^{(n+1)}(c_{n+1})$$

Now let $c = c_{n+1}$. □

By Fundamental Theorem of Calculus and Integration by Parts, we also have:

Theorem 6.6.3. If f has $n+1$-th order derivative and the $n+1$-th order derivative is continuous on $(-r, r)$, then

$$R_n = \int_0^x \frac{f^{(n+1)}(t)(x-t)^n}{n!} dt$$

□

6.7 Further Topics in Analysis

- Arzela-Ascoli, Stone-Weierstrass etc.
- **Measure Theory**: Lebesgue integration, Rietz representation, probability
- **Complex analysis**
- **Harmonic analysis**
- **Functional Analysis:** Gelfand's spectral theory, C^* algebra, von-Neumann algebra etc.
- **Ergodic Theory**
- **PDE**
- ...

Appendix A Midterm Review

Topics that will be covered in Midterm:

- $\mathbb{R}$
  - Smallest Upper Bound Property
  - Denseness of $\mathbb{Q}$ in $\mathbb{R}$, and related results.
  - Compactness of finite closed intervals, and related results.
- Metric Spaces
  - Open sets, closed sets
  - Continuity
  - Compactness
  - Open and closed sets in $\mathbb{R}$, intermediate value theorem

Exercises:

1: Let (X, d) be a metric space, $f, g : X \to \mathbb{R}$ continuous, show that $h : X \to \mathbb{R}$ defined by $h(x) = \max(f(x), g(x))$ is also continuous.

Answer: Can use ϵ-δ, or just by definition: any open subset of $\mathbb{R}$ is a union of finite open intervals of the form (a, b), $a < b$, and

$$h^{-1}((a,b)) = f^{-1}((-\infty, b)) \cap g^{-1}((-\infty, b)) \cap (f^{-1}((a,\infty)) \cup g^{-1}((a,\infty))$$

hence must be open in X.

2: Let (X, d) be a compact metric space, then for any $r > 0$, there is a finite subset $A_r \subseteq X$ such that for any $x \in X$, there is some $a \in A_r$ such that $d(x, a) < r$.

Answer: $X = \bigcup_{x\in X} B_X(x, r)$, hence compactness implies that there are finitely many $x_1, \dots, x_n$ such that $X = \bigcup_{i=1}^n B_X(x_i, r)$. $A_r = \{x_1, \dots, x_n\}$.

3: Let (X, d) be a metric space, V_n, $n \in \mathbb{N}$ non-empty compact subsets of X, $V_{n+1} \subseteq V_n$ for all n. Show that $\bigcap_{n\in\mathbb{N}} V_n$ is non-empty.

Answer: For any $n > 0$, because V_n is compact, $V_0 \backslash V_n$ is an open subset of V_0. If $\bigcap_{n\in\mathbb{N}} V_n = \emptyset$, then $V_0 = \bigcup_{n=1}^\infty (V_0 \backslash V_n)$, hence compactness of V_0 implies that there must be finite many $n_1, \dots n_m$ such that $V_0 = \bigcup_{j=1}^m (V_0 \backslash V_{n_j}) = V_0 \backslash V_{\max(n_1,\dots,n_m)}$. Hence $V_{\max(n_1,\dots,n_m)} = \emptyset$, a contradiction.

4: Let (X, d) be a compact metric space, $f : \mathbb{R} \to X$ a continuous map. Show that for any $\epsilon > 0$, for any $T > 0$, there are $x, y \in \mathbb{R}$, $y - x > T$, and

$d(f(x), f(y)) < \epsilon$.

Answer: Suppose not, then there is some $\epsilon > 0$, $T \geq 0$, such that for any $x, y \in \mathbb{R}$, if $d(f(x), f(y)) < \epsilon$ then $|x - y| \leq T$. Let $Y = \overline{f(\mathbb{R})}$, then Y is closed hence compact. The set $\{B_Y(f(t), \epsilon) : t \in \mathbb{R}\}$ is an open cover of Y, hence has a finite subcover, i.e. $Y = \bigcup_{i=1}^{n} B_Y(f(t_i), \epsilon)$. Let $t' = \max\{t_i\} + T + 1$, then $d(f(t'), f(t_i)) < \epsilon$ for some t_i yet $|t' - t_i| \geq T + 1 > T$, a contradiction.

Appendix B Practice Problems for Midterm

1. Let A be a non-empty closed subset of $\mathbb{R}$. Define function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = \sup\{r \in \mathbb{R} : r \leq 0 \text{ or } (x - 2r, x + r) \cap A = \emptyset\}$$

- Show that $g(x) = 0$ iff $x \in A$.
- Show that g is continuous.

Answer:

- If $x \in A$, for any $r > 0$, $A \cap (x - 2r, x + r) \neq \emptyset$, hence $g(x) = 0$.

  If $x \notin A$, closedness of A implies that there is $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \cap A = \emptyset$, hence $g(x) \geq \epsilon/2 > 0$.

- We only need to show that x is continuous at every point. If $x \in A$, for any $\epsilon > 0$, let $\delta = \epsilon$, then for any x' within δ-distance to x, the interval $(x' - 2r, x' + r)$ must have non-empty intersection with A for all $r \geq |x - x'|$, and $|x - x'| < \epsilon$. Hence $0 \leq g(x') < \epsilon$.

  If $x \notin A$, suppose $g(x) = y > 0$, then $(x - 2y, x + y) \cap A = \emptyset$. Furthermore, if $(a, b) \cap A = \emptyset$ and $a < x - 2y$, $b > x + y$, let $y' = \min(b - x, (x - a)/2)$, then $y' > y$ and $(x - 2y', x + y) \cap A = \emptyset$, which contradicts with the definition of g. Hence there can not be $a < x - 2y$, $b > x + y$ such that $(a, b) \cap A = \emptyset$.

  Given $\epsilon > 0$, let $\delta = \min(y/2, \epsilon/2)$. Then for any $x' \in (x - \delta, x + \delta)$, the argument in the previous paragraph implies that $g(x) - \epsilon < \max(y/2, g(x) - \epsilon/2) \leq g(x') \leq g(x) + \epsilon/2 < g(x) + \epsilon$.

2.

- Let (X, d) be a compact metric space, show that if $A \subset X$ is the intersection of an open set and a closed set in X, then for any $p \in A$, there is some $r > 0$ such that the intersection between A and the closed ball centered at p is compact.
- Find a subset of $\mathbb{R}$ which is not the intersection of an open set and a closed set.

Answer:

- Suppose $A = U \cap V$, U is open and V is closed. For any $p \in A$, let $r' > 0$ be such that $B_X(p, r') \subseteq U$, and let $r = r'/2$, then the closed ball B centered at p with radius r is a subset of U, hence $A \cap B = V \cap B$ is closed hence compact.

- The previous argument shows that for example $\mathbb{Q}$ is not the intersection of an open set and a closed set.

3. Let f be a function from $[0,1]$ to $[0,1]$, such that for any $c \in [0,1]$, $f^{-1}([0,c])$ is closed.

- Show that the set $\{(x,y) : 0 \leq x \leq 1, f(x) \leq y \leq 1\}$ is compact as a subspace of $\mathbb{R}^2$.
- Show that there is some $x \in [0,1]$ such that for any $x' \in [0,1]$, $f(x) \leq f(x')$.

Answer:

- Because $f^{-1}([0,c])$ is closed, its complement in $[0,1]$, $f^{-1}((c,1])$, is an open subset of $[0,1]$. Hence the set $U = \bigcup_{c\in(0,1)}\{(x,y) \in [0,1] \times [0,1] : y < c, x \in f^{-1}((c,1])\}$ is open. The set in question is $[0,1]\times[0,1]\backslash U$ hence closed hence compact.
- The map $(x,y) \mapsto y$ is continuous, hence there must be some point (x_m, y_m) in the set $\{(x,y) : 0 \leq x \leq 1, f(x) \leq y \leq 1\}$ where the value of this function is minimized. It is evident that y_m has to be $f(x_m)$.

Appendix C Final Review

Topics that will be covered in final exam:

- $\mathbb{R}$
  - Smallest Upper Bound Property
  - Denseness of $\mathbb{Q}$ in $\mathbb{R}$, and related results.
  - Compactness of finite closed intervals, and related results.
- Metric Spaces
  - Open sets, closed sets
  - Continuity
  - Compactness
  - Open and closed sets in $\mathbb{R}$, intermediate value theorem.
- Sequential Limit in metric space
  - Relationship between sequential limit and open set, closed sets and continuity
  - Cauchy sequences
  - Completeness, compact sets are complete.
  - Uniform convergence theorems
- Sequences and series
  - Monotone convergence, upper and lower limits
  - Comparison test, convergence of power series
  - Summation by parts
- Differentiation and integration
  - Definition of differentiation and Riemann integrals
  - Integrability of continuous functions, continuity implies uniform continuity on compact sets
  - Intermediate value theorem for derivatives, mean value theorem
  - Fundamental Theorem of Calculus
  - Uniform convergence, differentiation and integration
- L'Hospital's Rule and discrete L'Hospital's Rule.

Exercises:

1. Are the following numbers greater than 1, equals 1 or less than 1?

- $\lim_{N\to\infty} \frac{\sum_{n=1}^N \sin(\frac{1}{n^2})}{\sum_{n=1}^N \frac{1}{n^2}}$

- $\lim_{N\to\infty} \frac{\sum_{n=1}^N \sin(\frac{1}{n})}{\sum_{n=1}^N \frac{1}{n}}$

Answer: Because when $x > 0$, $\sin(x) < x$, we have

$$\sum_{n=1}^{\infty} \sin(\frac{1}{n^2}) < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

hence

$$\lim_{N\to\infty} \frac{\sum_{n=1}^N \sin(\frac{1}{n^2})}{\sum_{n=1}^N \frac{1}{n^2}} < 1$$

By discrete L'Hospital's Law,

$$\lim_{N\to\infty} \frac{\sum_{n=1}^N \sin(\frac{1}{n})}{\sum_{n=1}^N \frac{1}{n}} = \lim_{N\to\infty} \frac{\sin(\frac{1}{N})}{\frac{1}{N}} = 1$$

2. Let X be a metric space, show that $\lim_{n\to\infty} x_n = x$ iff the function $f_n : X \to \mathbb{R}$ defined by $y \mapsto d(y, x_n)$ converges uniformly to $g : X \to \mathbb{R}$ defined by $y \mapsto d(y, x)$.

Answer: For any $\epsilon > 0$, let $N \in \mathbb{N}$ such that $n > N$ implies $d(x_n, x) < \epsilon$, then for any $y \in X$,

$$|f_n(y) - g(y)| = |d(y, x_n) - d(y, x)| \leq d(x_n, x) < \epsilon$$

This shows that f_n converges to g uniformly.

3. Show that the function $f(x) = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}$ is differentiable at 0.

Answer: For any $\epsilon > 0$, let $\delta = \epsilon$, then for any $x \in (0 - \delta, 0 + \delta)$,

$$|f(x) - f(0) - 0(x - 0)| \leq |x^2| < \epsilon|x|$$

This shows that $f'(0) = 0$.

4. Let $\{a_n\}$ be a sequence of real numbers. Suppose $\lim_{n\to\infty} a_{2n} = 1$ and $\lim_{n\to\infty} a_{2n+1} = 0$.

- Show that $\lim_{n\to\infty} a_n$ does not exist.
- Calculate $\limsup_{n\to\infty} a_n$.
- Calculate $\lim_{n\to\infty} |a_n - a_{n+1}|$.
- Show that the set $\{a_n : n \in \mathbb{N}\} \cup \{0, 1\}$ is compact.

Answer:

- This is because if a sequence converges then any subsequence must converge to the same limit.

- The answer is 1. To show that, for any $\epsilon > 0$, find N_1 such that $n > N_1$ implies $a_{2n} \in (1-\epsilon, 1+\epsilon)$, and N_2 such that $n > N_2$ implies $a_{2n+1} \in (-\epsilon, \epsilon)$. Now for any $n > \max(2N_1, 2N_2+1)$, for any $k \geq n$, $a_k < 1+\epsilon$, so $\limsup_{n\to\infty} a_n \leq 1+\epsilon$. On the other hand, if $k \geq n$ and k is even then $a_k > 1-\epsilon$, so $\limsup_{n\to\infty} a_n > 1-\epsilon$. Because ϵ can be any positive real number, $\limsup_{n\to\infty} = 1$.

- This is 1, argument similar to the calculation of lim sup above.

- Denote this set as X, let C be an open cover, then there must be two elements $U_0, U_1 \in C$ which covers 0 and 1 respectively. By assumption $X\backslash(U_0 \cup U_1)$ consists of finitely many points hence can be covered by finitely many elements of C.

5. Suppose $\limsup_{n\to\infty} a_n = A$ and $\limsup_{n\to\infty} b_n = B$, show that $\limsup_{n\to\infty}(a_n + b_n) \leq A + B$.

Answer: For any $\epsilon > 0$, let N_1 satisfies that $n > N_1$ implies $\sup\{a_k : k \geq n\} < A + \epsilon$, N_2 satisfies that $n > N_2$ implies $\sup\{b_k : k \geq n\} < B + \epsilon$, then when $n > N = \max(N_1, N_2)$, the set $\{a_k + b_k : k \geq n\}$ has an upper bound $\sup\{a_k : k \geq n\} + \sup\{b_k : k \geq n\} < A + B + 2\epsilon$. This shows that $\limsup_{n\to\infty}(a_n + b_n) < A + B + 2\epsilon$. Because ϵ can be any positive real number, $\limsup_{n\to\infty}(a_n + b_n) \leq A + B$.

6. Show that any monotone increasing function on $[0, 1]$ is Riemann integrable on $[0, 1]$.

Answer: For any $\epsilon > 0$, let $n \in \mathbb{N}$ satisfies $(f(1) - f(0))/n < \epsilon$, let P be the partition $(0, 1/n, \ldots, (n-1)/n, 1)$, then

$$U(f,P)-L(f,P) = \sum_{i=1}^{n} \frac{1}{n} \sup(f([(i-1)/n, i/n])) - \sum_{i=1}^{n} \frac{1}{n} \inf(f([(i-1)/n, i/n]))$$

$$= \frac{1}{n}(\sum_{i=1}^{n} f(i/n) - \sum_{i=1}^{n} f((i-1)/n) = (f(1) - f(0))/n < \epsilon$$

7. Show that if $\sum_{n=0}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, 1]$. (Hint: using summation by parts)

Answer: We just need to show this sequence is uniformly Cauchy. Because the sequence of partial sums of a_n converges, it is Cauchy, in other words, for any $\epsilon > 0$, there is $N \in \mathbb{N}$, such that for any $q > p > N$, $|\sum_{n=p+1}^{q} a_n| < \epsilon/2$.

Now for any $q > p > N$, $x \in [0,1]$

$$|\sum_{n=p+1}^{q} a_n x^n| = |\sum_{n=p+1}^{q-1} (\sum_{k=p+1}^{n} a_k)(x^n - x^{n+1}) + (\sum_{n=p+1}^{q} a_n)x^q|$$

$$\leq \epsilon/2(\sum_{n=p+1}^{q-1} |x^n - x^{n+1}| + x^q) \leq \epsilon/2 < \epsilon$$

This shows that the partial sum sequence of $a_n x^n$ is uniformly Cauchy. Due to completeness of $\mathbb{R}$ this shows that the series converge uniformly.

8. Show that if f is bounded on $[a,b]$ and continuous on (a,b) then f is Riemann integrable on $[a,b]$.

Answer: Let M be an upper bound of $|f|$. For any $\epsilon > 0$, let $0 < r < \min((b-a)/4, \epsilon/(8M))$, and let $P = (x_0, \ldots, x_n)$ be a partition of $[a+r, b-r]$ such that $U(f,P) - L(f,P) < \epsilon/2$. Let P' be the partition $(a, x_0, \ldots, x_n, b)$, then

$$U(f,P') - L(f,P') \leq U(f,P) - L(f,P) + 2r \cdot 2M < \epsilon$$

Appendix D Review Problems for Final Exam

1. Let A and B be two non-empty subsets of $\mathbb{R}$, both A and B has upper bounds. Show that $\sup(A) \leq \sup(B)$ iff for any $\epsilon > 0$, for any $a \in A$, there is some $b \in B$ such that $b > a - \epsilon$.

Answer: $\Longrightarrow$: because $\sup(B)$ is the smallest upper bound of B, and $\sup(B) - \epsilon < \sup(B)$, there is some $b \in B$ such that $b > \sup(B) - \epsilon \geq \sup(A) - \epsilon \geq a - \epsilon$.

$\Longleftarrow$: if $\sup(B) < \sup(A)$, then let $\epsilon = (\sup(A) - \sup(B))/3$, $a \in A$ such that $a > \sup(A) - \epsilon$, then there can not be any $b \in B$ such that $b > a - \epsilon$.

2. Let (X, d) be a compact metric space with infinitely many elements, $r > 0$, show that there is some natural number N, such that any subset of X with N elements would have two element with distance less than r.

Answer: By compactness, X can be covered by finitely many open $r/3$-balls. Let M be the number of these $r/3$ balls and $N = M + 1$, then any N points in X must have at least 2 in the same $r/3$ ball, hence within distance $2r/3 < r$ of one another.

3. Let $\{a_n\}$ be a sequence of real numbers. Suppose $\sum_{n=0}^{\infty} a_n$ converges, show that $f_n(x) = \sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$ for any $0 < r < 1$.

Answer: This is because $\sum_{n=0}^{\infty} a_n$ converges implies $|a_n|$ is bounded, hence $\limsup_{n \to \infty} |a_n|^{1/n} \leq 1$.

4. Let g be a Riemann integrable function on $[-2, 2]$. Show that the sequence of functions: $f_n : [-1, 1] \to \mathbb{R}$ defined by

$$f_n(x) = 2^n \int_{x-2^{-n-1}}^{x+2^{-n-1}} g(t)dt$$

are all continuous, and when g is continuous, f_n converges uniformly to $g|_{[-1,1]}$.

Answer: Let M be the upper bound of $|g|$, then for any $x < x'$,

$$|f_n(x) - f_n(x')| \leq 2^n (|\int_{x-2^{-n-1}}^{x'-2^{-n-1}} g(t)dt| + |\int_{x+2^{-n-1}}^{x'+2^{-n-1}} g(t)dt|) \leq 2^{n+1} M |x' - x|$$

So f_n are all Lispchitz hence continuous.

If g is continuous it is also absolutely continuous. For any $\epsilon > 0$, let δ be such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \epsilon/2$, and let N be such that $2^{-N-1} < \delta$. Then for $n > N$, for any $x \in [-1, 1]$,

$$|f_n(x) - g(x)| \leq 2^n \int_{x-2^{-n-1}}^{x+2^{-n-1}} |g(t) - g(x)|dt \leq \epsilon/2 < \epsilon$$

5. Show that the set of functions that are differentiable at 0 is not closed under d_{sup} as a subset of the set of real valued continuous functions on $[-1, 1]$.

Answer: $f_n(x) = |x|^{(n+2)/(n+1)}$ are all differentiable at 0 but this sequence converges uniformly to $|x|$ which is not differentiable at 0.

Appendix E HW Solutions

E.1 HW 1

1. Let $\leq$ be a linear order on a set A. Let $B = A \times A$, define a relation $\leq'$ from B to itself as $(a, b) \leq' (c, d)$ if and only if either of the following two cases is true:

1. $a < c$
2. $a = c$ and $b \leq d$

Here, by $a < c$, we mean $a \leq c$ and $a \neq c$. Show that $\leq'$ is also a linear order.

Answer: Recall that a relation $\preceq$ between a set and itself is called a linear order if it satisfies:

1. $a \preceq a$
2. $a \preceq b, b \preceq a \implies a = b$
3. $a \preceq b, b \preceq c \implies a \preceq c$
4. $a \preceq b$ or $b \preceq a$

Now we check these four conditions for $\leq'$:

- Condition 1: For any $(a, b) \in B$, we have $a = a$. Also, because $\leq$ satisfies Condition 1, $b \leq b$. Hence by the definition of $\leq'$ we have $(a, b) \leq' (a, b)$.
- Condition 2: Suppose $(a, b) \leq' (c, d)$ and $(c, d) \leq' (a, b)$, by definition of $\leq'$ we have $a \leq c$ and $c \leq a$, hence $a = c$, which implies that $b \leq d$ and $d \leq b$, which implies $b = d$.
- Condition 3: Suppose $(a, b) \leq' (c, d)$ and $(c, d) \leq' (e, f)$. Then by definition of $\leq'$ we must be in one of the following four cases:
  - Case 1: $a = c = e$. In this case the definition of $\leq'$ implies that $b \leq d$ and $d \leq f$, hence $b \leq f$, we have $(a, b) \leq' (e, f)$.
  - Case 2: $a < c$ and $c < e$. This implies that $a \leq c$ and $c \leq e$, apply Condition 3 to $\leq$ we get $a \leq e$. On the other hand, if $a = e$ then $a \leq c$ and $c \leq a$ which implies $a = c$, which contradicts with the assumption that $a < c$. Hence $a \neq e$, which together with $a \leq e$ implies $a < e$. Hence by the definition of $\leq'$ we have $(a, b) \leq' (e, f)$.
  - Case 3: $a = c$, $c < e$. This implies $a < e$, hence $(a, b) \leq' (e, f)$.
  - Case 4: $a < c$, $c = e$. This implies $a < e$, hence $(a, b) \leq' (e, f)$.
- Condition 4: Suppose $(a, b) \not\leq' (c, d)$, we need to show that $(c, d) \leq' (a, b)$. By definition of $\leq'$, if $(a, b) \not\leq' (c, d)$, then $a \not< c$, which implies $c \leq a$. If $c < a$ then $(c, d) \leq' (a, b)$. If $c = a$, then $(a, b) \not\leq' (c, d)$ implies $b \not\leq d$, hence $d < b$, which implies $(c, d) \leq' (a, b)$.

2. Write down an injection from $\mathbb{Q}$ to $\mathbb{N}$. Let F be the set of injections from $\mathbb{Q}$ to $\mathbb{N}$, show that F has the same cardinality as the power set of $\mathbb{N}$. (Hint: You only need to show $|F| \leq |P(\mathbb{N})|$ and $|P(\mathbb{N}| \leq |F|$, by constructing injections between these two sets.)

Answer: An injection i from $\mathbb{Q}$ to $\mathbb{N}$ can be defined as follows: any element in $\mathbb{Q}$ can be uniquely written as $\pm p/q$ where $p, q \in \mathbb{N}$, $gcd(p, q) = 1$. Let

$$i(p/q) = 2^{p+1}3^{q+1}$$
$$i(-p/q) = 5^{p+1}3^{q+1}$$

The injectivity of i is due to unique factorization of integers.

An injection from F to $P(\mathbb{N})$ can be defined as follows:

$$f \mapsto \{2^{i(q)+1}3^{f(q)+1} : q \in \mathbb{Q}\}$$

And an injection from $P(\mathbb{N})$ to F can be defined as follows:

$$A \mapsto f_A$$

where

$$f_A(q) = \begin{cases} i(q) & q \notin A \\ 7i(q) & q \in A \end{cases}$$

The well definedness and injectivity of these maps also follows from unique factorization.

E.2 HW 2

1. Find a non-empty subset A of $\mathbb{R}$ that has an upper bound, and $\sup(A) \notin A$.

Answer: If $A = (0, 1)$ then $\sup(A) = 1 \notin A$.

2. Let $A \subseteq \mathbb{R}$, $A \neq \emptyset$. Show that a real number m is $\sup(A)$, if and only if it satisfies both of the following two conditions:

(1) For any $a \in A$, $a \leq m$

(2) For any $n \in \mathbb{N}$, $n > 0$, there is some number $b \in A$, such that $b > m - 1/n$.

Answer: Suppose $m = \sup(A)$, then m is an upper bound of A, hence (1) is true. Now we prove that (2) is true also. Suppose (2) is not true, then there is some positive natural number n such that for all $b \in A$, $b \leq m - 1/n$. Hence $m - 1/n$ is an upper bound of A that is less than m, a contradiction.

Suppose m satisfies both (1) and (2). (1) implies that m is an upper bound of A. Suppose there is another upper bound of A which is $m' < m$, let n be a natural number larger than $\frac{1}{m-m'}$, then $m - 1/n$ is an upper bound of A as well, which contradicts with (2). Hence m is also minimal among upper bounds of A.

E.3 HW 3

1. For any function $g : [0, \infty) \times [0, \infty) \to [0, \infty)$, let

$$d_g : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$$

be defined as

$$d_g((x, y), (x', y')) = g(|x - x'|, |y - y'|)$$

(1) Show that when $g(s, t) = s + \sqrt{t}$, d_g is a metric on $\mathbb{R}^2$.

(2) Write down a function g such that $(\mathbb{R}^2, d_g)$ is not a metric space. Justify your answer.

Answer:

(1)

$$\begin{aligned} d_g((x, y), (x', y')) &= g(|x - x'|, |y - y'|) \\ &= g(|x' - x|, |y' - y|) = d_g((x', y'), (x, y)) \end{aligned}$$

$$d_g((x, y), (x', y')) = 0 \iff |x - x'| + \sqrt{|y - y'|} = 0 \iff x = x', y = y'$$

Lastly,

$$\begin{aligned} & d_g((x, y), (x', y')) + d_g((x', y'), (x'', y'')) \\ &= (|x - x'| + |x' - x''|) + (\sqrt{|y - y'|} + \sqrt{|y' - y''|}) \\ &\geq |x - x''| + \sqrt{|y - y''|} = d_g((x, y), (x'', y'')) \end{aligned}$$

Here the $\geq$ is due to

$$\sqrt{a} + \sqrt{b} = \sqrt{(\sqrt{a} + \sqrt{b})^2} = \sqrt{a + b + 2\sqrt{a}\sqrt{b}} \geq \sqrt{a + b}$$

(2) Let $g = 0$, then $d_g((0, 0), (1, 1))$, hence d_g is not a metric.

2. Let $p : [0, 1] \to (0, 1)$ be a function.

(1) Show that there is a finite set $A \subset [0, 1]$, such that for any $x \in [0, 1]$, there is some $a \in A$ such that $|x - a| < p(a)$.

(2) Show that there is some $\epsilon > 0$, such that for any $x \in [0, 1]$, there is some $y \in [0, 1]$ such that $|x - y| + \epsilon < p(y)$.

Answer:

(1)

$$[0, 1] \subseteq \bigcup_{a \in [0,1]} (a - p(a), a + p(a))$$

Hence there must be finitely many elements $a_1, \ldots, a_n$ where

$$[0, 1] \subseteq \bigcup_{i=1}^{n} (a_i - p(a_i), a_i + p(a_i))$$

We can now let $A = \{a_1, \ldots, a_i\}$.

(2) Let A be as above, consider the finite set $B = \{a_i \pm p(a_i) : i \in \{1, \ldots, n\}\}$. Let r' be the shortest distance between two successive elements in B, $r = \min(r', 0 - \min(B), \max(B) - 1)$, then for any $x \in [0, 1]$, $(x - r/2, x + r/2)$ contains at most one element in B. Suppose it doesn't contain any element in B, then let y be an element $a_i \in A$ such that $x \in (a_i - p(a_i), a_i + p(a_i))$, we have

$$(x - r/2, x + r/2) \subseteq (a_i - p(a_i), a_i + p(a_i))$$

$$|x - a_i| + r/3 < p(a_i)$$

Suppose it does contain an element in B, this element must also be in $[0, 1]$ hence must be in some $(a_i - p(a_i), a_i + p(a_i))$, and the end points of this open interval $(a_i - p(a_i), a_i + p(a_i))$ can not be in $(x - r/2, x + r/2)$. Hence we also have

$$(x - r/2, x + r/2) \subseteq (a_i - p(a_i), a_i + p(a_i))$$

$$|x - a_i| + r/3 < p(a_i)$$

Alternative solution for 2(2): Because $[0, 1] \subseteq \bigcup_{x \in [0,1]} (x - p(x)/2, x + p(x)/2)$, by compactness there must be finitely many elements $x_1, \ldots x_k$ in $[0, 1]$ such that for any $y \in [0, 1]$, there is some x_i such that $|y - x_i| < p(x_i)/2$. Hence we can let $\epsilon = \min(p(x_1)/3, \ldots, p(x_k)/3)$.

3. Let $X = \{[a, b] : a, b \in \mathbb{R}, a < b\}$ be the set of all finite closed intervals in $\mathbb{R}$. Define $d : X \times X \to [0, \infty)$ as

$$d([a, b], [a', b']) = \inf(\{t \in \mathbb{R} : t \geq 0, [a, b] \subseteq [a' - t, b' + t], [a', b'] \subseteq [a - t, b + t]\})$$

Show that (X, d) is a metric space. You need to check that d is well defined first.

Answer:

- First check that d is well defined: the set $\{t \in \mathbb{R} : t \geq 0, [a, b] \subseteq [a' - t, b' + t], [a', b'] \subseteq [a - t, b + t]\}$ is non-empty because $|a| + |b| + |a'| + |b'| + 1$ is always in it, and has a lower bound 0, hence must always have an infimum.
- The symmetry is obvious from the definition of d.
- It is evident that $0 \in \{t \in \mathbb{R} : t \geq 0, [a, b] \subseteq [a - t, b + t], [a, b] \subseteq [a - t, b + t]\}$, hence $d([a, b], [a, b]) = 0$. On the other hand, if $[a, b] \neq [a', b']$, then one of the endpoints of one of these two closed intervals can not be in the other. Without loss of generality assume that $a \notin [a', b']$, then $\min(|a - a'|, |a - b'|) > 0$ is a lower bound of elements in $\{t \in \mathbb{R} : t \geq 0, [a, b] \subseteq [a' - t, b' + t], [a', b'] \subseteq [a - t, b + t]\}$, hence $d([a, b], [a', b']) \neq 0$.
- Suppose $d([a, b], [a', b']) = d_1$, $d([a', b'], [a'', b'']) = d_2$. Then, for any $\epsilon > 0$, there is some $d_1' < d_1 + \epsilon$ such that

$$[a, b] \subseteq [a' - d_1', b' + d_1']$$

$$[a', b'] \subseteq [a - d_1', b + d_1']$$

Similarly, there is some $d_2' < d_2 + \epsilon$ such that

$$[a', b'] \subseteq [a'' - d_2', b'' + d_2']$$

$$[a'', b''] \subseteq [a' - d_2', b' + d_2']$$

Hence

$$[a, b] \subseteq [a'' - d_1' - d_2', b'' + d_1' + d_2']$$
$$[a'', b''] \subseteq [a - d_1' - d_2', b + d_1' + d_2']$$

In other words

$$d([a, b], [a'', b'']) \leq d_1 + d_2 + 2\epsilon$$

Because ϵ can be arbitrarily small we have

$$d([a, b], [a'', b'']) \leq d_1 + d_2$$

which is the triangle inequality.

E.4 HW 4

1. Let (X, d) be a metric space. For any subset $A \subseteq X$, we define $\overline{A}$ as the intersection of all closed sets in X that have A as a subset, and A° as the union of all subsets of A which are also open subsets of X.

(1) Show that a point $p \in X$ is in A°, if and only if there is some $r > 0$, such that the open ball centered at p with radius r, $B_X(p, r)$, is a subset of A.

(2) Show that a point $p \in X$ is in $\overline{A}$, if and only if for any $r > 0$, $B_X(p, r) \cap A \neq \emptyset$.

(3) Show that

$$A^\circ \subseteq (\overline{(A^\circ)})^\circ \subseteq \overline{((\overline{A})^\circ)} \subseteq \overline{A}$$

(4) In the case when X is $\mathbb{R}$ with Euclidean metric, find an $A \subseteq \mathbb{R}$ such that

$$A^\circ \subsetneq (\overline{(A^\circ)})^\circ \subsetneq \overline{((\overline{A})^\circ)} \subsetneq \overline{A}$$

Answer:

(1) $p \in A^\circ$ iff there is some open set U, $p \in U$ and $U \subseteq A$. If there is such a U, let $r > 0$ satisfies $B_X(p, r) \subseteq U$, then $B_X(p, r) \subseteq A$; if there is some $r > 0$ such that $B_X(p, r) \subseteq A$, we can let $U = B_X(p, r)$, then U is open, $p \in U$ and $U \subseteq A$.

(2) $p \in \overline{A}$ iff for any closed set V, iff for any open set $U \subseteq X \backslash A$, $p \notin U$ iff $p \notin (X \backslash A)^\circ$, which by the previous question, is equivalent to the fact that for any $r > 0$, $B_X(p, r) \not\subseteq X \backslash A$, or in other words, $B_X(p, r) \cap A \neq \emptyset$.

(3) A° is an open set and $A^\circ \subseteq \overline{A^\circ}$, hence $A^\circ \subseteq (\overline{A^\circ})^\circ$. Because $A \subseteq \overline{A}$, any subset of A which is open must also be a subset of $\overline{A}$, hence $A^\circ \subseteq (\overline{A})^\circ$, similarly we have $\overline{A^\circ} \subseteq \overline{(\overline{A})^\circ}$, hence $(\overline{A^\circ})\circ \subseteq \overline{A^\circ} \subseteq \overline{(\overline{A})^\circ}$. Lastly, $\overline{A}$ is closed and $(\overline{A})^\circ \subseteq \overline{A}$, hence $\overline{(\overline{A})^\circ} \subseteq \overline{A}$

(4) $A = (0,1) \cup (1,2) \cup ([2,3] \cap \mathbb{Q}) \cup \{4\}$. Then the four sets are $(0,1) \cup (1,2)$, $(0,2)$, $[0,3]$ and $[0,3] \cup \{4\}$ respectively.

2. Let (X,d) be a metric space, $d' : X \times X \to [0,\infty)$ is defined as

$$d'(p,q) = \min(1, d(p,q))$$

Show that the identity map id_X is a continuous function from (X,d') to (X,d).

Answer: For any $\epsilon > 0$, let $\delta = \min(\epsilon, 1/2)$, then $d'(x,y) < \delta$ implies $d(x,y) < \epsilon$, hence id_X as a function from (X,d') to (X,d) is continuous.

E.5 HW 5

Unless specified otherwise, the metric on $\mathbb{R}$ or any of its subset are assumed to be the Euclidean metric $d(x,y) = |x-y|$.

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, and the range $f(\mathbb{R})$ is a subset of $\mathbb{Q}$. Show that f must be a constant function. (Hint: use intermediate value theorem).

Answer: Suppose not, there are two real numbers $a < b$ such that $f(a) \neq f(b)$, hence by intermediate value theorem there is some $c \in (a,b)$ such that $f(c) = f(a) + (f(b) - f(a))/\sqrt{2} \notin \mathbb{Q}$, which is a contradiction.

2. Suppose f and g are two continuous functions from $\mathbb{R}$ to $\mathbb{R}$. $f(q) = g(q)$ for all $q \in \mathbb{Q}$. Show that $f = g$.

Answer: Because both f and g are continuous, so is $f-g$. Suppose $f-g \neq 0$, there must be some $x \in \mathbb{R}$ where $f(x) \neq 0$, hence by continuity there must be some $r > 0$ such that $f(x') \neq 0$ for all $x' \in (x-r, x+r)$. However denseness of $\mathbb{Q}$ in $\mathbb{R}$ implies that $(x-r, x+r)$ must contain rational numbers, a contradiction.

3. Let $f : (-\infty, 0] \to \mathbb{R}$ and $g : [0,\infty) \to \mathbb{R}$ be two continuous functions, and $f(0) = g(0)$. Show that the function $f : \mathbb{R} \to \mathbb{R}$

$$h(x) = \begin{cases} f(x) & x \leq 0 \\ g(x) & x > 0 \end{cases}$$

is continuous.

Answer: For any $\epsilon > 0$, because f is continuous, there is $\delta_1 > 0$ such that for any $x' \in B_{(-\infty,0]}(0, \delta_1) = (-\delta_1, 0]$, $|f(x') - f(x)| < \epsilon$. Similarly, there is some $\delta_2 > 0$ such that for any $x' \in [0, \delta_2)$, $|g(x') - g(0)| < \epsilon$. Hence, if we let $\delta = \min(\delta_1, \delta_2) > 0$, then as long as $|x' - x| < \delta$, $|h(x') - h(x)| < \epsilon$. Hence h is continuous at 0. Continuity of f also implies continuity of h on $(-\infty, 0)$, while continuity of g implies continuity of h on $(0, \infty)$, hence h is continuous.

4. Let (X, d) be a metric space, $A \subseteq X$, $A \neq \emptyset$.

(1) Show that the function $h_A : X \to \mathbb{R}$ defined by $h_A(x) = \inf(\{d(x, a) : a \in A\})$ is continuous.

(2) Show that $\overline{A} = \{x \in X : h_A(x) = 0\}$.

(3) If A is compact under the subspace metric, then for any $x \in X$ there is some $a \in A$ such that $d(a, x) = h_A(x)$.

Answer:

(1) h_A is a 1-Lipschitz function, because for any $x, x' \in X$,

$$h_A(x) - h_A(x') = \inf(\{d(x, a) : a \in A\}) - \inf(\{d(x', a) : a \in A\})$$

$$\leq \inf(\{d(x', a) + d(x', x) : a \in A\}) - \inf(\{d(x', a) : a \in A\}) = d(x', x)$$

Switching x and x', we know that $|h_A(x) - h_A(x')| \leq d(x, x')$.

(2) It is evident that $A \subseteq h_A^{-1}(\{0\})$, and continuity of f_A implies that $h_A^{-1}(\{0\})$ is closed, hence $\overline{A} \subseteq h_A^{-1}(\{0\})$. On the other hand, if $x \notin \overline{A}$, because $X \backslash \overline{A}$ is open, there is some $r > 0$ such that $A \cap B_X(x, r) = \emptyset$, hence $h_A(x) \geq r > 0$. This shows $h_A^{-1}(\{0\}) \subseteq \overline{A}$.

(3) The function $d(\cdot, x)$ restricted to a compact set A is continuous, hence must reach its minimum at some $a \in A$.

5. Suppose (X, d) is a compact metric space, $A \subseteq X$, and A is an infinite set. Show that there is some $x \in X$, such that for any $\epsilon > 0$, the intersection between A and the open ball in X centered at x with radius ϵ is an infinite set.

Answer: Suppose not, for every $x \in X$, there is some $\epsilon_x > 0$ such that $A \cap B_X(x, \epsilon_x)$ is a finite set. $X = \bigcup_{x \in X} B_X(x, \epsilon_x)$, hence by compactness of X, there are finitely many $x_1, \ldots, x_n$ such that $X = \bigcup_{i=1}^n B_X(x_i, \epsilon_{x_i})$, hence $A = \bigcup_{i=1}^n (A \cap B_X(x_i, \epsilon_{x_i}))$ which is a finite union of finite sets, a contradiction.

E.6 HW 6

Let (X, d) be a metric space of diameter 1. Let $Z = Map(\mathbb{N}, X)$ be the set of sequences on X. Let $d_{sup} : Z \times Z \to [0, \infty)$ be defined as

$$d_{sup}(a, b) = \sup\{d(a(n), b(n)) : n \in \mathbb{N}\}$$

1. Prove that (Z, d_{sup}) is a metric space.

Answer: Because X is bounded, any function from $\mathbb{N}$ to X is a bounded function, hence (Z, d_{sup}) is a metric space.

2. For any $x \in X$, show that the set L_x consisting of all sequences whose limit is x, is closed in Z.

Answer: For any $a \notin L_x$, there is ϵ such that for any $N \in \mathbb{N}$, there is some $n_N > N$ such that $d(a(n_N), x) \geq \epsilon$. Consider the open ball centered at a with radius $\epsilon/2$ under metric d_{sup}, then for any b in this open ball, for any $N \in \mathbb{N}$, we have $n_N > N$ and

$$d(b(n_N), x) \geq d(a(n_N), x) - d(a(n_N), b(n_N)) > \epsilon/2$$

hence $b \notin L_x$. This shows that the compliment of L_x is open, hence L_x is closed.

3. Show the set C consisting of all Cauchy sequences on X is closed in Z.

Answer: For any $a \notin C$, there is some $\epsilon > 0$ such that for any $N \in \mathbb{N}$, there is $m_N > n_N > N$ such that $d(a(m_N), a(n_N)) \geq \epsilon$. Now for any $b \in B_{(Z,d_{sup})}(a, \epsilon/3)$, for any $N \in \mathbb{N}$,

$$d(b(m_N), b(n_N)) \geq d(a(m_N), a(n_N)) - d(a(m_N), b(m_N)) - d(a(n_N), b(n_N)) \geq \epsilon/3$$

hence $b \notin C$. This shows that the compliment of C is open, hence C is closed.

4. Show via an counter example, that the set L consisting of convergent sequences on X is not necessarily closed.

Let $X = \mathbb{Q}$ and d be the Euclidean metric restricted to $\mathbb{Q}$. Let $q = \{q_n\}$ be a sequence of rational numbers convergent to $\sqrt{2}$ in $\mathbb{R}$, then $q \notin L$. Let p_j be defined as

$$p_j(n) = \begin{cases} q_n & n \leq j \\ p_j & n > j \end{cases}$$

Then the limit of p_j is $q_j \in \mathbb{Q}$, hence $p_j \in L$. On the other hand $\lim_{j\to\infty} p_j = q$, which implies that L is not closed.

E.7 HW 7

1. Let $\{a_n\}$ be a sequence of real numbers, $|a_n| \leq 1$ for all n. Show that

- $\limsup_{n\to\infty} a_n \geq \liminf_{n\to\infty} a_n$.
- $\{a_n\}$ converges in $\mathbb{R}$ iff $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$.

Answer:

- By definition of lim sup and lim inf, For any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that when $n > N$,

$$\sup\{a_j : j \geq n\} < \limsup_{n\to\infty} a_n + \epsilon$$

And there is $N' \in \mathbb{N}$ such that when $n > N'$,

$$\inf\{a_j : j \geq n\} > \liminf_{n\to\infty} a_n - \epsilon$$

Let $N'' = \max\{N, N'\} + 1$, then

$$\liminf_{n\to\infty} a_n - \epsilon < \inf\{a_j : j \geq n\} \leq a_{N''} \leq \sup\{a_j : j \geq N''\} < \limsup_{n\to\infty} a_n + \epsilon$$

Hence

$$\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n > -2\epsilon$$

Because ϵ can be any positive real number,

$$\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n \geq 0$$

- Suppose $\lim_{n\to\infty} a_n = b$, then given any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that when $n > N$, $b - \epsilon < a_n < b + \epsilon$. Hence

$$\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n \leq \sup\{a_j : j \geq N+1\} - \inf\{a_j : j \geq N+1\}$$

$$\leq (b+\epsilon) - (b-\epsilon) = 2\epsilon$$

Because ϵ can be any positive real number, this is only possible when $\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n = 0$.

On the other hand, if

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = b$$

then for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $n > N$ implies

$$\sup\{a_j : j \geq n\} < b + \epsilon$$

And there is $N' \in \mathbb{N}$ such that when $n > N'$,

$$\inf\{a_j : j \geq n\} > b - \epsilon$$

Hence for any $n > \max\{N, N'\}$,

$$b - \epsilon < \inf\{a_j : j \geq n\} \leq a_n \leq \sup\{a_j : j \geq n\} < b + \epsilon$$

which implies $\lim_{n\to\infty} a_n = b$.

2. Let f_n be a sequence of functions from $[0,1]$ to $\mathbb{R}$, that converges uniformly to $g : [0,1] \to \mathbb{R}$. Show that if for all n, f_n is a polynomial of degree at most 1 (i.e. of the form $x \mapsto ax + b$ for some $a, b \in \mathbb{R}$), then g is also a polynomial of degree at most 1.

Answer: Because f_n is a polynomial of degree at most 1,

$$f_n(x) = (1-x)f_n(0) + xf_n(1)$$

Because $\{f_n\}$ converges uniformly to g, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $n > N$ implies

$$|f_n(0) - g(0)| < \epsilon$$

and there is $N' \in \mathbb{N}$ such that $n > N'$ implies

$$|f_n(1) - g(1)| < \epsilon$$

Hence for any $n > \max\{N, N'\}$, any $x \in [0,1]$

$$|f_n(x) - (g(0)(1-x) + g(1)x)| \leq |f_n(0) - g(0)|(1-x) + |f_n(1) - g(1)|x < \epsilon$$

Hence f_n uniformly converges to $g(0)(1-x) + g(1)x$ which is a polynomial of degree no more than 1.

3. Let (X, d) be a compact metric space. $F : X \times X \to X$ a continuous map from $(X \times X, d_{sup})$ to (X, d). Suppose $\{a_n\}$ is a sequence in X that converges to b, show that the sequence of functions $f_n : X \to X$ defined as $x \mapsto F(x, a_n)$ converges uniformly to $f_b : X \to X$ defined as $x \mapsto F(x, b)$. (Hint: use an argument similar to the proof that the product of two compact metric spaces are compact under d_{sup}.)

Answer: For any $x \in X$, continuity of F implies that for any $\epsilon > 0$, there is $r_{x,\epsilon} > 0$ such that $F(B_{X\times X}((x,b), r_{x,\epsilon})) \subseteq B_X(F(x,b), \epsilon/2)$. As a consequence, if $d(a,b) < r_{x,\epsilon}$ and $d(x', x) < r_{x,\epsilon}$, then $d(F(x',a), F(x',b)) < \epsilon$. By compactness of X, there is a finite set $\{x_1, \dots, x_n\} \subseteq X$ such that $\bigcup_{i=1}^n B_X(x_i, r_{x_i,\epsilon}) = X$. Let $r_\epsilon = \min\{r_{x_i,\epsilon}\} > 0$, then as long as $d(a,b) < r_\epsilon$ we always have $d(F(x,a), F(x,b)) < \epsilon$ for all $x \in X$. Let $N \in \mathbb{N}$ such that for any $n > N$, $d(a_n, b) < r_\epsilon$, then for any $n > N$, any $x \in X$, $d(f_n(x), f_b(x)) < \epsilon$, which proved that f_n uniformly converges to f_b.

E.8 HW 8

1. Let $\{a_n\}$ be a sequence of real numbers, $\{|a_n|^{\frac{1}{n}}\}$ is unbounded. Show that for any $x \neq 0$, the series $\sum_{n=0}^\infty a_n x^n$ diverges.

Answer: For any $x \neq 0$, suppose $\sum_{n=0}^\infty a_n x^n$ converges, then $\lim_{n\to\infty} |a_n x^n| = 0$, hence there is some $N \in \mathbb{N}$ such that for any $n > N$, $|a_n x^n| < 1$, hence $|a_n|^{\frac{1}{n}} < \frac{1}{|x|}$ when $n > N$. This implies that the set $\{|a_n|^{\frac{1}{n}}\}$ has an upper bound

$\max(\max\{|a_n|^{\frac{1}{n}} : n \leq N\}, \frac{1}{|x|})$, a contradiction.

2. Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences of positive real numbers, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, $b_{n+1} < b_n$ for all n, and $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = 1$. Is it true that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ as well? Prove it or find a counter example.

Answer: This is true. For any $\epsilon > 0$, pick $N \in \mathbb{N}$ such that when $n > N$,

$$\left|\frac{a_{n+1}-a_n}{b_{n+1}-b_n} - 1\right| < \frac{\epsilon}{2}$$

In other words

$$|a_{n+1} - a_n - (b_{n+1} - b_n)| < \frac{\epsilon}{2}(b_n - b_{n+1})$$

For any $n > N$, pick $n' > n$ such that $|a_{n'} - b_{n'}| < \frac{\epsilon b_n}{2}$, then

$$\left|\frac{a_n}{b_n} - 1\right| \leq \left|\frac{\sum_{j=n}^{n'-1} a_j - a_{j+1} - (b_j - b_{j+1})}{b_n}\right| + \left|\frac{a_{n'} - b_{n'}}{b_n}\right| < \epsilon$$

3. Find a real number A such that $A \leq \sum_{n=0}^{\infty} \frac{\sin(n)}{n+1} \leq A+1$. You can use calculator but need to justify your answer mathematically.

Answer: Let $a_n = \sin(n)$, $b_n = \frac{1}{n+1}$, then

$$s_n = \sum_{i=0}^{n} a_n = \sum_{i=0}^{n} \frac{\cos(n-1/2) - \cos(n+1/2)}{2\sin(1/2)}$$

So $|s_n| \leq \frac{1}{\sin(1/2)}$.

For any natural number m, if $n >> m$ then

$$|\sum_{i=0}^{n} a_i b_i - \sum_{i=0}^{m} a_i b_i| = |\sum_{i=m+1}^{n} a_i b_i| = |\sum_{i=m+1}^{n} (s_i - s_{i-1}) b_i|$$

$$= |-s_m b_{m+1} + \sum_{i=m+1}^{n-1} s_i(b_i - b_{i+1}) + s_n b_n| \leq \frac{1}{\sin(1/2)}|b_{m+1}| = \frac{1}{(m+2)\sin(1/2)}$$

So

$$|\sum_{i=0}^{n} a_i b_i - \sum_{i=0}^{3} a_i b_i| < 1/2$$

We can let $A = \sum_{i=0}^{3} \frac{\sin(i)}{i+1} - \frac{1}{2} \approx 0.2591$

E.9 HW 9

1. Write down a function from $\mathbb{R}$ to $\mathbb{R}$ which is differentiable on $[0,\infty)$ but not differentiable on $(-\infty,0)$.

Answer: $f(x)=\begin{cases}x^2 & x\geq 0 \text{ or } x\in\mathbb{Q}\\ 0 & \text{otherwise}\end{cases}$.

2. Show that if a function f from $\mathbb{R}$ to $\mathbb{R}$ is differentiable on $\mathbb{R}$ and a bijection, then either $f'(x)\geq 0$ for all $x\in\mathbb{R}$ or $f'(x)\leq 0$ for all $x\in\mathbb{R}$.

Answer: If otherwise, there are two real numbers $a\neq b$ where $f'(a)<0$ and $f'(b)>0$. Suppose $a<b$, then there is some $\epsilon>0$, $\epsilon<b-a$ such that $f(a+\epsilon)<f(a)$, $f(b+\epsilon)>f(b)$. If $f(a)>f(b)$, then apply intermediate value theorem to $[a,b]$ and $[b,b+\epsilon]$ for some m that satisfies

$$f(b)<m<\min(f(a),f(b)+\epsilon)$$

we get that f can not be injective. If $f(a)<f(b)$, apply intermediate value theorem to $[a,a+\epsilon]$ and $[a+\epsilon,b]$ for some m that satisfies

$$f(a+\epsilon)<m<f(a)$$

we get that f can not be injective. The case when $a>b$ is analogous.

3. Let $f:\mathbb{R}\to\mathbb{R}$ be a function that is differentiable on $\mathbb{R}$, and f' is 1-Lipschitz as a function from $\mathbb{R}$ to $\mathbb{R}$. Show that the sequence of functions

$$g_n(x)=n(f(x+1/n)-f(x))$$

converges uniformly to f' as $n\to\infty$.

Answer: By mean value theorem, $g_n(x)=f'(c_x)$ where $c_x\in(x,x+1/n)$. Because f' is 1-Lipschitz,

$$|g_n(x)-f'(x)|=|f'(c_x)-f'(x)|\leq|c_x-x|<\frac{1}{n}$$

4. Let $f:[0,1]\to\mathbb{R}$ be defined as

$$f(x)=\begin{cases}1 & x=2^{-n}, n\in\mathbb{N}\\ 0 & \text{otherwise}\end{cases}$$

Show that f is Riemann integrable on $[0,1]$.

Answer: For any $\epsilon>0$, pick N such that $2^{-N}<\epsilon$, consider the partition

$$P_N=(0,2^{-N-1},2^{-N}-\frac{2^{-N-1}}{2N+4},2^{-N}+\frac{2^{-N-1}}{2N+4},2^{-N+1}-\frac{2^{-N-1}}{2N+4},2^{-N+1}+\frac{2^{-N-1}}{2N+4},\dots,$$

$$1-\frac{2^{-N-1}}{2N+4},1)$$

Then $L(f,P_N)=0$ and

$$U(f,P_N)=2^{-N-1}+\frac{2^{-N-1}}{2N+4}+\sum_{i=1}^{N}\frac{2^{-N-1}}{N+2}<2^N<\epsilon$$

E.10 HW 10

1. f is a real valued differentiable function on $(0,1)$. Show that if f is unbounded then f' is also unbounded.

Answer: For any $M>0$, there is some $x\in(0,1)$ such that $|f(x)-f(1/2)|>M$. Hence by mean value theorem, there is some c between $1/2$ and x such that $|f'(c)|>2M$.

2. f is a bounded real valued function on $[-1,1]$, and is continuous on both $[-1,0)$ and $(0,1]$. Show that f is Riemann integrable on $[-1,1]$.

Answer: Let M be an upper bound of $|f|$. For any $\epsilon>0$, find partition $P=(x_0,\dots,x_n)$ of $[-1,-\frac{\epsilon}{8M}]$ such that $U(f,P)-L(f,P)<\epsilon/4$, and partition $P'=(x'_0,\dots,x'_{n'})$ of $[\frac{\epsilon}{8M},1]$ such that $U(f,P')-L(f,P')<\epsilon/4$, then under partition $P''=(x_0,\dots,x_n,x'_0,\dots,x'_{n'})$ of $[-1,1]$,

$$\begin{aligned}U(f,P'')-L(f,P'')&=(U(f,P)-L(f,P))+(U(f,P')-L(f,P'))\\&+\frac{\epsilon}{4M}(\sup(f([-\frac{\epsilon}{8M},\frac{\epsilon}{8M}]))-\inf(f([-\frac{\epsilon}{8M},\frac{\epsilon}{8M}])))\\&<\epsilon/4+\epsilon/4+\epsilon/2=\epsilon\end{aligned}$$

This shows that f is Riemann integrable on $[-1,1]$.

3. Write down a sequence of real valued functions $\{f_n\}$ on $[0,1]$ which converges uniformly to another function g, such that f_n are all differentiable at $1/2$ but g is not differentiable at $1/2$.

Answer: For example, $f_n(x)=\max(|x-1/2|-2^{-n},0)$, and $g(x)=|x-1/2|$.

E.11 Honors HW

1. Let (X,d) be a metric space. Show that X is compact iff any sequence has a convergent subsequence.

Answer: $\implies$ is covered in class. To show $\impliedby$, suppose any sequence on X has a convergent subsequence, we shall prove that X is compact.

Firstly, for any $r > 0$, X can be covered by finitely many closed r-balls. If that's not true, pick a_0 as an arbitrary point on X, for each positive natural number n let a_n be a point not in the union of closed r-balls centered at $a_0, \dots, a_{n-1}$, then this produces a sequence with no convergent subsequence, a contradiction.

Now suppose $\mathcal{C}$ is an open cover of X without a finite subcover. For any natural number n, let $\mathcal{B}_n$ be a finite collection of closed 2^{-n} balls that cover X, then at least one of them can not be covered by any finite subset of $\mathcal{C}$. Let b_n be the center of this closed 2^{-n}-ball. Suppose $\{b_n\}$ has a subsequence with limit $b \in X$, then there must be some $U \in \mathcal{C}$ that contains b, hence some ϵ-ball centered at b for some $\epsilon > 0$ can be covered by a single element in $\mathcal{C}$. Pick $N >> 1$ such that $d(b_N, b) < \epsilon/2$, $2^{-N} < \epsilon/2$, then the closed 2^{-N}-ball centered at b_N can be covered this single element, a contradiction.

2. Consider the set $Map([0,1],[0,1])$ under d_{sup}, let p be a function from $(0,\infty)$ to $(0,\infty)$.

(1) Let

$$C_p = \{f \in Map([0,1],[0,1]) : \text{ for all } \epsilon > 0, \text{ for all } x, x' \in [0,1],$$
$$\text{if } |x - x'| < p(\epsilon) \text{ then } |f(x) - f(x')| \leq \epsilon\}$$

Show that C_p is closed in $Map([0,1],[0,1])$ under d_{sup} metric.

(2) Let $\{f_n\}$ be a sequence in C_p. Show that if for any $q \in \mathbb{Q} \cap [0,1]$, $\lim_{n\to\infty} f_n(q)$ exists, then $\{f_n\}$ converges uniformly.

(3) Show that any sequence in C_p has a uniformly convergent subsequence. In other words, C_p is compact.

Remark: The third part of problem 2 is called the **Arzela-Ascoli Theorem**.

Answer:

(1) Suppose $f \notin C_p$, there is some $\epsilon > 0$ such that there are two $x, x' \in [0,1]$ such that $|x - x'| < p(\epsilon)$ and $|f(x) - f(x')| > \epsilon$. Let $\epsilon' = \frac{|f(x)-f(x')|-\epsilon}{3} > 0$ then the ϵ'-ball centered at f would be disjoint from C_p.

(2) For any $\epsilon > 0$, let M be a natural number large enough such that $1/M < p(\epsilon/3)$. Now for any $j = 0, 1, \dots, M$, pick natural number N_j large enough, such that for any $m, n > N_j$, $|f_n(j/M) - f_m(j/M)| < \epsilon/3$, and let $N = \max(N_0, \dots, N_M)$.Then for any $m, n > N$, any $x \in [0,1]$, there is some natural number j such that $|x - j/M| < p(\epsilon/3)$, hence

$$|f_m(x)-f_n(x)| \leq |f_m(x)-f_m(j/M)|+|f_m(j/M)-f_n(j/M)|+|f_n(x)-f_n(j/M)| < \epsilon$$

This shows that $\{f_n\}$ is uniformly Cauchy hence it converges uniformly.

(3) For any sequence $\{f_n\}$ on C_p, let $i : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$ be a bijection, pick a subsequence $\{f_{n_{0,i}}\}$ that converges at $i(0)$, then pick another subsequence of the first subsequence, denoted as $\{f_{n_{1,i}}\}$, which converges at $i(1)$, and continue for all natural numbers. Now $\{f_{n_{i,i}}\}$ is a subsequence that converges at all points in $\mathbb{Q} \cap [0, 1]$, hence converges uniformly on $[0, 1]$.