Math 300: Introduction to Mathematical Reasoning

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Purpose of the course

- Introduction of the language of mathematics (mathematical logic and how to read and write proofs)
- Introduction of basic mathematical concepts, including numbers (natural numbers, integers, rationals and reals) and sets.
- This course is essential for any serious study of mathematics and might be helpful for students of other subjects as well.

Grading

Grading: 40% final, 25% each midterm, 5% homework, 5% workshop
Topics covered in this course: logic and proofs, theory of natural numbers, theory of sets, theory of real numbers
Homework and exams will be based on lecture notes, because we likely won't be able to cover the whole textbook.

- A or B: Understand some basic strategies for writing proofs, ready to take further math courses.
- C: Able to read proofs, need further work during the summer in order to take other math courses.
- D: Others.

Topics we will cover (each taking about a week)

Proposition logic and truth table*
Predicate logic and the rules of deduction*
Definition of natural numbers
Mathematical induction
Addition and multiplication
Divisibility and primes
Definition of sets
Products, relations and functions
Equivalent classes
Integers
Rationals and reals
*: We will come back to these topics now and then throughout the semester.

Further reading

If you are like things you see in this course, you may be interested in:

- Learning more mathematics. You should be ready for the introductory text of any branch of mathematics (analysis, algebra, topology, combinatorics etc.)
- Learning more about logic or theoretical computer science (a popular science book on these subjects is Hofstadter's *Gödel, Escher, Bach*, and you should be ready for any introductory texts on these subjects too)

Things to pay attention to while taking this course

- Trust nothing but your own reason.
- Make sure you understand every steps and details.
- Practice.
- Try applying what you learned to other courses you're taking, as well as to everyday life.

An example of proof

Theorem: $\sqrt{2}$ is not a rational number.
Proof: We need to show that there aren't integers m and n such that $m^2/n^2 = 2$. Suppose m and n are such integers. Then $m^2 = 2n^2$. Factorize m and n into primes, we have $m = \prod_p p^{c_p}$, $n = \prod_p p^{d_p}$, and $m^2 = 2n^2$ implies $2c_p = 2d_p$ if $p \neq 2$ and $2c_2 = 1 + 2d_2$, the latter is impossible because c_2 and d_2 must both be integers.

Propositional Logic

Propositions: sentences which can be either true or false. In propositional logic, propositions consists of either single symbols like A, B, C, ... (atomic propositions) or symbols connected by $\vee$, $\wedge$, $\neg$ ($\sim$ in textbook), $\implies$, $\impliedby$, $\iff$, and parenthesis when needed.
The truth value of composite propositions can be calculated using the truth values of its components. For example, $A \wedge B$ is true if both A and B are true, and false if otherwise.
A proposition is a tautology if its value is always true for all possible truth values on the single symbols it contains. For example, $((A \implies B) \wedge A) \implies B$.

How to find tautologies

To check when a statement is a tautology, one can either:

- Substitute the symbols it contains with all possible truth values (truth table)
- Deduce it from known tautologies using a set of deduction rules.

More examples of tautologies: $\neg(A \wedge \neg A)$, $A \implies (A \vee B)$, $(A \implies B) \iff (\neg A \vee (A \wedge B))$.

A set of deduction rules for proposition logic

- If there is a sequence of deduction from A to B, then $A \implies B$.
- From $A \implies B$ and A one can get B.
- From A and B one can get $A \wedge B$.
- From $A \wedge B$ one can get either A or B.
- From either A or B one can get $A \vee B$.
- From $A \vee B$, $A \implies C$, and $B \implies C$ one can get C.
- From $A \iff B$ one gets $(A \implies B) \wedge (B \implies A)$ and vice versa.
- From $\neg\neg A$ one gets A and vice versa.
- If there is a sequence of deduction from A to Contradiction, then $\neg A$.
- From A and $\neg A$ one gets Contradiction
- From Contradiction one can get any proposition.

Review for Lecture I

- Some basic concepts in logic
- Concepts in proposition logic
- Deductions in proposition logic

Basic concepts about logic and deductions

- A **deduction** consists of a sequence of sentences or other deductions.
- There are two types of sentences: some are true or false, while others contain unknown quantities (free variables) and its truthness depends on the choice of those quantities. The first type are called **propositions**, the latter called **predicates**.
- When we add ("deduce") a proposition in the deduction, we mean that if all the preceding propositions in its context is true, then this new proposition is true. When we add a predicate, we mean if its free variables are chosen so that all the preceding sentences are true, then it is true.
- For example, from "if $1 > 0$, then $0 > -1$" and "$1 > 0$" we can deduce "$0 > -1$". From "if $a > 0$ then $0 > -a$" and "$a > 0$" we can deduce "$0 > -a$".

Concepts in Proposition logic

- A sentence in proposition logic consists of letters, which represent simple propositions, connected by $\wedge$, $\vee$, $\implies$, $\iff$ and $\neg$.
- $A \wedge B$ is true if and only if both A and B are true. $A \vee B$ is false if and only if both A and B are false. $\neg A$ is true if and only if A is false. $A \implies B$ means $\neg A \vee (A \wedge B)$, and $A \iff B$ means $(A \implies B) \wedge (B \implies A)$.
- The kind of sentences we care about in proposition logic are **tautologies**, i.e. sentences whose value is always true no matter what the truth values of the letters it contains are.
- We can check if a sentence is a tautology by enumerating over all possible truth values for all the simple propositions it contains. This process is called the **truth table**.
- If a sentence in proposition logic is a tautology, replacing all letters with any propositions, the resulting proposition is always true hence can be used as a step in deduction.

Deduction in proposition logic

To save space, we use $\vdash$ to mean "deduce from", and $\perp$ to mean "contradiction".

- Rules governing $\wedge$: $A, B \vdash A \wedge B$; $A \wedge B \vdash A$; $A \wedge B \vdash B$.
- Rules governing $\vee$: $A \vdash A \vee B$; $B \vdash A \vee B$; $A \vee B, A \implies C, B \implies C \vdash C$.
- Rules governing $\implies$: $(A \vdash B) \vdash A \implies B$; $A, A \implies B \vdash B$.
- Rules governing $\iff$: $(A \implies B) \wedge (B \implies A) \vdash A \iff B$; $A \iff B \vdash (A \implies B) \wedge (B \implies A)$.
- Rules governing $\neg$: $A \vdash \neg\neg A$; $\neg\neg A \vdash A$.
- Rules about contradiction: $A, \neg A \vdash \perp$; $(A \vdash \perp) \vdash \neg A$; $\perp \vdash A$.
- The application of the second rule about contradiction is called **proof by contradiction**.

An example of deduction

Theorem: $A \wedge B \iff \neg(A \implies \neg B)$
Proof: Assume $A \wedge B$ is true. This implies both A and B are true. Further assume $A \implies \neg B$ is true, then, because A is true, we get $\neg B$, which contradicts with B, hence $\neg(A \implies \neg B)$ is true. This argument gives us $A \wedge B \implies \neg(A \implies \neg B)$.
Next, assume $\neg(A \implies \neg B)$ is true. Suppose $\neg A$, then if A is true, there is a contradiction, hence $\neg B$. This tell us $A \implies \neg B$, which contradicts with the initial assumption, hence A must be true. On the other hand, suppose $\neg B$, then if A we have $\neg B$, hence $A \implies \neg B$, which also contradicts with the initial assumption, hence B must be true also. Hence, $A \wedge B$. This argument gives us $\neg(A \implies \neg B) \implies A$, and, together with the previous paragraph, the theorem is proved.

Predicate logic

- In predicate logic, we allow upper case letters to represent predicates: e.g. $A(x, y)$ is a predicate with two free variables x and y, and use lower case letters to denote functions and constants.
- There are three more symbols: $\forall$, $\exists$ and $=$, which come together with more deduction rules (here t is any given term(formed by variables and functions)):
  - Rules governing $\exists$: $A(t) \vdash \exists x A(x)$; $\exists x A(x), A(y) \implies B \vdash B$.
  - Rules governing $\forall$: $A(x) \vdash \forall y A(y)$; $\forall x A(x) \vdash A(t)$.
  - Rules governing $=$: $\vdash t = t$; $s = t \vdash t = s$; $r = s, s = t \vdash r = t$.
  - Predicates and functions: $s = t \vdash f(s) = f(t)$; $s = t \vdash P(s) \iff P(t)$.
- The conclusions deduced from these rules are true statements in predicate logic, i.e. they are true when one replace the letters with any predicate or function.

An example of deduction in predicate logic

Theorem: $\neg\exists x P(x) \iff \forall x \neg P(x)$
Proof: Assume that $\neg\exists x P(x)$. Suppose for some y, $P(y)$, then $\exists x P(x)$, which contradicts with the initial assumption. Hence $\neg P(y)$ must be true. Since y is arbitrary (free) in the argument above, we must have $\forall x \neg P(x)$.
Now assume that $\forall x \neg P(x)$. Suppose $\exists x P(x)$. Then, pick y such that $P(y)$ is true. From the assumption $\forall x \neg P(x)$, we know $\neg P(y)$, which results in a contradiction. Hence $\exists x P(x)$ must be false, which implies that $\neg\exists x P(x)$ is true. Together with the previous paragraph, the theorem is proved.

Remarks about writing proofs

In practice, the steps in deductions that relies solely on first order logic (like the ones you see in this week) are usually omitted when one writes down mathematical proofs, and one should be able to tell via intuition if such a deduction is valid. However for the beginning portion of this semester you will be required to write down all the details including those involving logic as an exercise and also as a way to guarantee that our intuitions are indeed correct.

Remarks about writing proofs

The rules introduced here are called "natural deduction" as they are close to what people really do when reasoning in their everyday life. However the proofs written using these rules might still feel weird at times and there is occasion need for reorganizing the sentences a bit in order to make the proof more readable.
It is not necessary to memorize the 24 rules listed, just be aware of their existence, and understand their validity. With practice you would be able to internalize the rules of logic by the end of the semester.

Survey

Please answer the following questions in a piece or paper (you do not need to write your name):

- Do you know how to prove that there are infinitely many prime numbers?
- Are you able to follow the lecture at the moment?
- What are some concepts we covered so far that you are still confused about?
- Do you have any further questions or suggestions?