1 9/5 Matrices, vectors, and their applications

Algebra: study of objects and operations on them.

Linear algebra: object: matrices and vectors. operations: addition, multiplication etc.

Algorithms/Geometric intuition/sets and maps

$m \times n$ matrix: numbers forming a rectangular grid, m rows and n columns. Motivation: coefficients of a system of linear equations. Data tables in statistics.

(i, j)-th entry of a matrix.

Vectors: matrices with one row/column. Motivation: coordinates in plane and space.

Operations: (1) Addition. (2) Scalar multiplication. (3) Matrix-vector multiplication. (4) Transpose.

Example: $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Ax, $A(Ax)$.

Example: Averaging over columns. Covariance? Other statistical concepts?

Laws: The usual laws one may expect. e.g. $A(x + y) = Ax + Ay$, $(A + B)^T = A^T + B^T$, $(A^T)^T = A$. Note: $A(Bx) \neq B(Ax)$!

Zero and one matrix. Standard vectors.

Example: Rotation by 60 degrees (or $\pi/3$).

Consequence: Matrix is completely determined by its action on the standard vectors! Matrix-matrix multiplication.

Example: 2×2 case.
The concept of linear combination. Relationship with matrix-vector multiplication.

Example: Rotation and Translation.

Example: Random walk on graphs.

2 9/8 Linear equations

Review:

- Matrix multiplications
- Transposes
- Standard vectors
- Identity Matrix
- Rotation matrix

- Stochastic matrix

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Linear systems as matrix equations. *Coefficient matrix* and *augmented matrices*

Elementary row operations: swap, multiply, add. Property: reversible, and preserves solution set.

Row echelon form: The first non-zero entry (called pivot) of each row is to the right of the previous. *Reduced row echelon form*: The first non-zero entry is 1 and is the only non-zero entry in that column. Uniqueness under row operations.

Algorithm (Gaussian elimination):

- Write augmented matrix.
- Use row operations, turn it into reduced echelon form.
- General solution from RREF (Example: $x_1 + 2x_2 + x_3 + x_4 = 3$, $x_1 + 3x_3 - x_4 = 8$).

Structure of solutions:

	Pivot at last col.	No pivot at last col.
All coefficient col. have pivot	None	One
Some coeff. col. have no pivot	None	Inf

Examples of the 4 cases.

True or false:

- A system of 3 linear equations with 6 variables can not have just one solution.
- A system of 3 linear equations with 6 variables must have infinitely many solutions.

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Counting: number of arbitrary constants and the number of pivots. Rank and dimension.

Explicit algorithm from RREF to general solutions.

3 9/12 Linear equations cont.

3.1 Review

- Augmented matrix, row operations.
- RREF.
- Condition for no/one/infinitely many solutions.
- General solution: write *basic variables* in terms of *free variables*, or the *vector form*.

3.2 Gaussian elimination

Augment matrices to REF or RREF through finitely many elementary row operations.

For r=$1, 2, \ldots n$:

Find the left-most non-zero entry among the $r, r+1, \ldots n$ rows. If there aren't any, terminate.
Exchange rows to move this entry to the r-th row.

Multiply the r-th row and add it to the $r+1,\dots$ rows to eliminate all entries on the left-most non-zero column.

To Further turn it into a RREF (backward pass):

Multiply to each non-zero row to make the first entry 1.
For each non-zero row, multiply and add it to each of the rows above it to turn the entries on pivot columns 0.

Reason for distinguish forward/backward passes: forward pass is a permutation matrix with a lower triangular matrix with 1 on the diagonals, backward pass is a upper triangular matrix. Row pivoting.

Example: $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 2 & 4 & 7 \\ 2 & 0 & 1 & 0 \end{pmatrix}$. RREF? General solution?

3.3 Uniqueness of RREF

Key idea: read the RREF from matrix using linear combinations of rows or columns!
Appendix E uses columns. One can also use rows as follows: Let R be the space of linear combination (span) of the row vectors. The last non-zero row in RREF is the one in R with the most number of 0 entries on the left and the first non-zero entry 1. Let the index of the first non-zero entry be c_1. The preceding row in RREF is the one in R with c_1-th entry 0, first non-zero entry 1, and the most possible number of 0 on the left, etc.

3.4 Rank and nullity

Rank of A: num. of pivots in A=num. of non-zero rows in REF of A=num of basic variables in $Ax = b$
Nullity of A: num of non-pivot columns in A=num. of columns of A-rank of A=num of free variables in $Ax = b$

True or false:

- The rank of $[A \;\; B]$ must be no smaller than the sum of the ranks of A and B.
- The nullity of $[A \;\; B]$ must be no smaller than the sum of the ranks of A and B.
- The RREF of a square matrix of no nullity must be the identity matrix
- The nullity of A is non-zero iff some row of A is a linear combination of the others.
- $[A \;\; B]$ has the same rank as B iff the columns of A are linear combinations of the columns of B.

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Structure of the general solution in terms of rank or nullity:
If $rank(A) < rank([A, b])$:
  No solution.
 Else:
  If $nullity(A) = 0$:
   One solution.
  Else:
   Infinitely many solutions.

Example: $\begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix}$.

4 9/15 Span

Review:

- Augmented matrix and row operations
- REF, RREF, pivot
- free and basic variables
- Rank and Nullity

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Linear combination: S is a set of matrices of the same size, v is called a linear combination of S iff there exist finitely many matrices $A_1 \dots A_n$ in S, and scalars $a_1, \dots a_n$, so that $v = \sum_k a_k A_k$.

Span: The span of a set is the set of all linear combinations of that set. S is called a generating set of the set $Span(S)$.

Example: span of the standard vectors.

Span closed under addition and scalar multiplication.

Transitivity.

b is in the span of columns of A iff $Ax = b$ has a solution.
$\mathcal{R}^n$: all vectors of n entries. Span is $\mathcal{R}^n$ iff matrix is *full rank* iff ...

Example: use linear equation to detect spans.

Implication on the rank of the matrices while adding columns.

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Algorithm for minimal generating set. Example.

True or false:

Row operation changes the span of the column vectors.

A matrix is in REF, then the span of the columns are the span of some standard vectors.

5 9/18 Linear dependency

5.1 Review

Notation: when A and B has the same number of rows, by $[A \ \ B]$ we mean a larger matrix formed by stacking them together horizontally.

relationship between matrices, system of equations $Ax = b$, and the column vectors:

The followings are equivalent:

- $Ax = b$ has a solution (is **consistent**).
- b lies in the **span** of the columns vectors of A.

- The **span** of the columns of A is the same as the span of the columns of A and b.
- **rank**$([A\ \ b]) = rank(A)$.
- **Nullity**$([A\ \ b]) = Nullity(A) + 1$.
- In the **RREF** of $[A\ \ b]$, the last column does not contain a **pivot**.

Examples.

The followings are equivalent:

- $Ax = b$ has a solution (is **consistent**) for all b.
- The **span** of the columns vectors of A is $\mathcal{R}^m$.
- **rank**$(A) = m$.
- **Nullity**$(A) = n - m$.
- In the **RREF** of A, every row contain a **pivot**.
- The **RREF** of A does not contain zero rows.

Examples.

5.2 Linear dependence/independence

A set S is called **linearly independent**, if for any sequence of distinct elements $x_1, \ldots x_k \in S$, $c_1x_1 + \ldots c_kx_k = 0$ implies that $c_1 = c_2 = \cdots = 0$. If a set is not linearly independent it is linearly dependent.

$a_1 \ldots a_n$ are linearly dependent if and only if $[a_1 \ldots a_n]x = 0$ (the **homogeneous eq.**) has one (hence infinitely many) non-zero solutions. (hence $[a_1 \ldots a_n]x = b$ has infinitely many solutions for some b, hence has free variables, hence the nullity of A is non-zero).

Example: 1 or 2 vectors.

Linear dependency in standard vectors.

Linear dependency in RREF.

Linear dependency in vector form of the general solution.

5.3 Number of rows and columns

$m > n$: column vectors may or may not be linearly dependent, but can never span $\mathcal{R}^m$.

$m < n$: column vectors may or may not span $\mathcal{R}^m$, but can never be linearly independent.

$m = n$: column vectors span $\mathcal{R}^m$ iff they are linearly independent.

5.4 Adding and removing vectors

If S is linearly independent, any subset of S is linearly independent and has a smaller span, $S \cap \{v\}$ is linearly independent iff v is in the span of S.

If S is linearly dependent, so is any set larger than S.

Examples.

******Optional*******

Row vectors under row operation.

Rank=num. of linearly independent column vectors.

Vertical stacks of matrices.

Relationship between homogeneous and non-homogeneous equations.

6 9/22 Review of Chapter 1, more on matrix multiplication

Important concepts to remember:

- Matrix
  - Identity matrix
  - Zero matrix
  - Scalar multiplication
  - Addition
  - Linear combination
  - Span
  - Linear independence
  - Transpose
  - Symmetric matrix
  - Row operation
  - REF, RREF
  - Pivot
  - Rank
  - Nullity
- Vector
  - Standard vectors
  - $\mathcal{R}^n$
- System of linear equation
  - Homogeneous equation

- consistence
- Augmented matrix
- Coefficient matrix
- Free variable
- Basic variable
- General solution
- General solution in vector form

True or false:
A set of 3 vectors in $\mathcal{R}^3$ is either linearly dependent or spans $\mathcal{R}^3$.
If the nullity of A is greater than 0, then $Ax = b$ has infinitely many solutions.
If $Ax = b$ has a unique solution, then the nullity of the augmented matrix is 1.

Fibonacci series.

Unique circle passing through 3 points.

$x + ay = b$, $cx + dy = e$.

7 9/26 Matrix algebra

Review:

Relationship between homogeneous and non-homogeneous system: if $Ax = b$ is consistent, x_0 is a solution, then any solution can be written as $x_0 + x_1$ where x_1 is a solution of $Ax = 0$.

Finding minimal generating set: Put into matrix, find pivot columns.

Matrix multiplication: three equivalent ways of defining it:

- row-column rule
- multiple matrix-vector multiplication
- composition: $AB = [A(Be_1), A(Be_2), \ldots]$.

Example using 2-by-2 matrices
Properties: the usual one, except

- No longer commutative.
- relationship with transposes.

Multiplication by identity matrix and diagonal matrix.
Example: matrix algebra and complex numbers. The idea of linear representation.

Example: non-commutativity of 3-d rotation.

8 9/29 Elementary matrix, inverses

Elementary row operation is left-multiplication by elementary matrices.

Column correspondence property.

Applications:

- Read linear relation from RREF.
- General solution of homogeneous equations.
- Row operation doesn't change solution.
- General solution of non-homogeneous equation.
- Uniqueness of RREF

Definition of the Inverse of a matrix. Elementary matrices are invertable.

9 10/3 Invertibility

A matrix is invertable iff $rank = \#rows = \#columns$.

Algorithm for A^{-1}.

Algorithm for solving $AX = B$.

Solution of $Ax = b$ when A is invertable: $x = A^{-1}b$.

10 Midterm I review

The midterm exam will cover up to section 2.3. Please make sure you know the following:

- Able to calculate the product between matrices and vectors.
- Able to solve system of linear equations with Gaussian elimination.
- Know the meaning of the following terms: matrix, identity matrix, zero matrix, symmetric matrix, diagonal matrix, elementary matrix, transpose, linear combination, span, linear dependency, row operation, pivot, rank, nullity, free variable, basic variable.
- Able to translate statements about matrices to statements about linear equations and vice versa. For example, the columns of matrix A are linear independent, then $Ax = 0$ has a unique solution.

Practice problems:

(1) Find t so that the vectors $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ t \end{bmatrix}$ are linearly dependent.

Solution: This is asking when $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & t \end{bmatrix}$ has rank smaller than 3. By Gaussian elimination you can see that it has rank smaller than 3 iff $t = 5/2$.

(2) True or false:

a) Elementary row operations does not change the span of column vectors.

b) If $Ax = b$ has at least two solutions, then the column vectors of A are linearly dependent.

Solution: a) is false. For example, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ can be turned into $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by exchanging the two rows, and the span of the columns are not the same. b) is true, because $Ax = b$ has more than one solution means that $Ax = 0$ has non-zero solution, hence the columns of A are linearly dependent by the definition of matrix-vector multiplication.

(3) Find matrix E so that $E\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution: You can solve a system of linear equation, or alternatively, recognize that to turn $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ into $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ one can first add the first row to the second then exchange the two rows, hence $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(4) Show that the transpose of elementary matrices are also elementary matrices.

Solution: write down the three types of elementary matrices and see that their transposes are elementary matrices of the same type.

11 10/13 Block multiplication cont. Linear transformation and matrices

11.1 Block multiplication

AB can be calculated by dividing A and B into rectangular blocks so that the block numbers and the sizes of blocks matches.

Example: Divide A and B into row/column vectors.

11.2 Linear transformation

Linear transformation: A map T from R^n to R^m is a linear transformation iff $T(x+y) = T(x) + T(y)$, $T(ax) = aT(x)$.

A is a $m \times n$ matrix, then $x \mapsto Ax$ is a linear transformation.

Write down the matrix of linear transformations: $[T(e_1), \dots T(e_n)]$.

Composition, identity, surjectivity, injectivity, and inverse.

Examples.

Translation between the 3 viewpoints:

Equations¡—¿Matrices¡—¿Spaces and maps

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Real-life application of block multiplication: BLUP in linear mixed models.

Determinant in 2×2.

12 10/17 Determinant

Recall: det: Defined on square matrices. the simplest poly. which characterizes invertability. Geometrically related to volumns.

Definition: A square matrix, A_{ij} A with i-th row, j-th column removed, $c_{ij} = (-1)^{i+j} det(A_{ij})$ the *cofactor*, then $\det([a]) = a$, $det(A) = \sum_i a_{1i} c_{1i}$.

Properties:

(1) Linear for each column.

(2) $det(I) = 1$.

(3) Negative when switching columns.

(4) Linear for each row.

(5) Negative when switching rows.

(6) Cofactor expansion for other rows/columns.

(7) Invariant under transposes.

(8) $det(EA) = det(E)det(A)$.

(9) $det(AB) = det(A)det(B)$.

(10) Inverse

(11) Cramer's law

(12) For diagonal matrices, the det is the product of entries on the diagonal

Proof: (1)

(1)-(5) by induction. (6): the case for columns follows from (1), (3) while the case for rows follows from the definition and (5). (7) follows from (6) and the definition, by induction. (8) follows from (4), (5). (9) follows from (8). (10) follows from (6) and (11) follows from (10) and (6). (12) follows immediately from the definition.

Examples: 2-by-2, 3-by-3, 4-by-4 computed using LU decomp.

13 10/20 Subspaces

Definition.

Example: span, null, col, row, generating set, kernel, range.

Relationship between null, col, row and matrix product and transposes.

Finding generator of null space: solving $Ax = 0$. Building matrix from a generating set of its null space.

14 10/24 Basis and Dimension

Definition of Basis.

Basis and linearly independent sets/generating sets.

Dimension.

Methods to find basis.

Application: $rank(A) = rank(A^T)$.

15 10/27 Basis of col, row, nul. Coordinates.

Basis of col, row, nul.

Monotonicity.

Application: Rank and matrix product.

Coordinate under a basis.

16 10/31 Coordinates, similarity, eigenvalues

B is a basis of V, then every element v in V can be written as a unique linear combination of elements of B, the coeff. vector is called the **coordinate of** v **under** B, denoted as $[v]_B = M_B^{-1}v$ where M_B is the matrix whose columns are the elements of B.

Example.

Usage: simplify equations, tensors.

$V = R^n$, then a basis B gives a map from R^n to R^n: $v \mapsto [v]_B$. Representation of a linear map T_M under coordinate B. Similarity.

Example.

Things that are unchanged under similarity: rank, nullity, identity matrix, eigenvalues.

Definition of eigenvalues, eigenvectors, and eigenspaces. Disjointness of eigenspaces.

Example of finding eigenvalues and eigenvectors: eigenvalues: roots of char. poly; eigenvectors: solve linear system. In practice: QR dec.

17 11/3 Eigenvalues and eigenspaces, charateristic polynomials

Review:

Similarity, eigenvalue, eigenspace, intersection of eigenspaces.

Example for calculation.

Examples: Rotation matrix, diagonal matrix, triangle matrix, symmetric matrix, antisymmetric matrix. Relationship between multiplicity of eigenvalue and dimension of eigenspaces.

Applications: Matrix power. Solving ODE and PDE. Physics. PCA.

Example: exp

True or false:
$A^2 = I$ then the eigenvalues of A must be ± 1.
Eigenvectors of A are also eigenvectors of A^T.
Eigenspaces are invariant under similarity.

18 11/7 Diagonalization

Theorem 1: The eigenvectors from different eigenvalues are linearly independent.

Theorem 2: The matrix A can be written in the form PDP^{-1} iff there is a basis of A consisting of eigenvectors.

Example for calculating diagonalization.

Checking diagonalization: 1. roots and multiplicity; 2. dim=mult.

Application: power, exp, roots, Caylay-Hamilton.

Jordan normal form: example.

Symmetric matrices.

19 11/10 Applications, dot product and norm

Review:

Eigenvalue, Eigenspace
Diagonalization, $A = P^{-1}DP$ then $f(A) = P^{-1}f(D)P$ for f polynomial or power series.

Stochastic matrix and Markov chain, pagerank
Stochastic matrix then 1 is an eigenvalue.
Regular (products non-zero) then the corresponding eigenvector is unique. (Peron-Frobenius, need some calculus).

A diagonalizable, $BA = AB$ iff B preserves eigenspaces of A. A matrix commutes with all diagonal matrices iff it's diagonal itself, a matrix commutes with all matrices iff it's a multiple of the identity.

Linear difference relation, example

Linear ODE

20 11/14 Dot products and norms

Definition of dot products, norm and angle. Basic properties.
Caychy-Schwarz ineq., triangle ineq.

21 Midterm 2 review

Multiplication by elementary matrices and row operations, LU decomposition.
Inverse: definition, computation, properties, invertability.
Determinant: cofactor expansion, computation, application (Cramer's rule)
Subspaces: definition, basis and dimension, row/col/nul space of a matrix, $rank(A) = rank(A^T)$
Eigenvalues and eigenspaces: similarity, definitions, linear independence of eigenvectors in different eigenspaces, diagonalization.

Review questions:

1. Find the characteristic polynomial of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$ and check if it is diagonalizable.

Solution: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 2 & 2-\lambda & 2 & 2 \\ 3 & 3 & 3-\lambda & 3 \\ 4 & 4 & 4 & 4-\lambda \end{vmatrix} = -2\begin{vmatrix} 1 & 1-\lambda/2 & 1 & 1 \\ 1-\lambda & 1 & 1 & 1 \\ 3 & 3 & 3-\lambda & 3 \\ 4 & 4 & 4 & 4-\lambda \end{vmatrix} =$
$-2\begin{vmatrix} 1 & 1-\lambda/2 & 1 & 1 \\ 0 & (3\lambda-\lambda^2)/2 & \lambda & \lambda \\ 0 & 3\lambda/2 & -\lambda & 0 \\ 0 & 2\lambda & 0 & -\lambda \end{vmatrix} = 2\lambda^3\begin{vmatrix} 1 & 1-\lambda/2 & 1 & 1 \\ 0 & 3/2 & -1 & 0 \\ 0 & (3-\lambda)/2 & 1 & 1 \\ 0 & 2 & 0 & -1 \end{vmatrix} = 2\lambda^3\begin{vmatrix} 1 & 1-\lambda/2 & 1 & 1 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 2-\lambda/3 & 1 \\ 0 & 0 & 4/3 & -1 \end{vmatrix} =$
$\lambda^3(\lambda - 10)$. Because the nullity of A is 3 and the nullity of $A - 10I$ is 1, it is diagonalizable.

2. True or false: if A is diagonalizable and invertable, then A^{-1} is diagonalizable.

Solution: True. $A = PDP^{-1}$ for some diagonal D and invertable P, hence $A^{-1} = PD^{-1}P^{-1}$.

3. True or false: A non-zero vector can not be in both the null and the column space of A.

Solution: False, for example, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

4. Find t so that $\left\{ \begin{bmatrix} 1 \\ 0 \\ t \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ t \\ 0 \\ -t \end{bmatrix} \right\}$ is a basis of the column space of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & t & t^2 \\ t & 0 & t^2 \\ 0 & -t & -t \end{bmatrix}$.

Solution: By Gaussian elimination, the column space is spanned by the first two columns if and only if $t^2 = t$ hence $t = 0$ or 1. $t = 1$ is the only case when the set given is linearly independent.

22 11/22 Orthogonal vectors

Review: dot product, norm, distance, cosine theorem, Cauchy Schwartz, triangular inequality.

Orthogonal. Orthogonal projection to a line in 2-dimension.

Orthogonal projection to a line in general.

Orthogonal set: nonzero then linearly indep.

Orthogonal/orthonormal basis: coordinates, GS, QR (matrices with nullity 0).

23 11/28 6.3 Orthogonal projection

Review: dot product, distance, orthogonlity, orthogonal basis, orthonormal basis, G-S.

Orthogonal to a set: check a basis of its span.

Geometric interpretation of GS: the concept of orthogonal projetion onto a subspace.

Definition. Uniqueness. Computation for orthonormal basis: formula for orthogonal basis, formula for general basis ($C(C^TC)^{-1}C^T$). ($C^T(x - C(C^TC)^{-1}C^Tx) = C^Tx - C^Tx = 0$). Examples.

G-S from the prespective of orthogonal projection: a 3×3 example.

Applications of QR: (1) computation of determinant (2) solving linear system in a more stable way (3) Find eigenvalues: $QR = RQ$ then...

minimizing distance.

Orthogonal complement. Row space as orth comp. of nul space.

24 11/1 Application of orthogonal projection

Review: Orthogonal projection, QR, orthogonal complement.
Inconsistent equations: linear least square: projection to col
Solution with the smallest norm: projection to nul.
least square approximation.
Orthogonal matrix: 2-by-2 example.

25 11/5 Orthogonal matrix, diagonalization of symmetric matrices

Review: orth. matrix, 2-by-2.

The followings are equivalent for square matrix A: A has orth. columns; $A^TA = I$; A preserves dot product; A preserves norm.

A orth then $|det(A)| = 1$, $A^T = A^{-1}$ orth, and if B is also orth then AB is orth.

Find orthogonal matrices with some given columns: find their orth complement, then do G-S.

Eigenspaces of symmetric matrices are orthogonal to each other.

The (complex) eigenvalues of a symmetric matrix is real.

Symmetric matrices are diagonalizable.

Proof: choose an orthnormal basis of the sum of all eigenspaces of M. Let C be orth with the first columns the basis elements, then C^TMC is symmetric of the form $\begin{bmatrix} D & 0 \\ 0 & M' \end{bmatrix}$.

Applications:

Spectral decomposition and spectrial approximation. PCA.

26 11/8 Application of orthogonal matrices

Quadratic form. Conic sections.

SVD.

Rotation in 3 dimension.

27 Final review

- Matrix multiplication, transpose, and inverse.
- Definition and calculation of determinant.
- Definition of subspace, span and basis, dimension, orthogonal complement, orthogonal and orthonormal basis.
- Gaussian elimination, elementary matrix, rank, col row and nul space
- Definition of similarity, characteristic polynomial, eigenspace and eigenspaces.
- Diagonalization, LU and QR decomposition of matrices, diagonalization of symmetric matrix, spectral decomposition.

1. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$, $B = A^T A$.
(1) Find an orthonormal basis for the rol, column and nul space of B.
(2) Find symmetric matrix C so that $B = C^2$.

Solution: (1) The row and column spaces are the same and via Gram-Schmit process an orthonormal basis is $\left\{ \begin{bmatrix} \sqrt{1/5} \\ \sqrt{1/5} \\ \sqrt{1/5} \\ \sqrt{1/5} \\ \sqrt{1/5} \end{bmatrix}, \begin{bmatrix} -\sqrt{2/5} \\ -\sqrt{1/10} \\ 0 \\ \sqrt{1/10} \\ \sqrt{2/5} \end{bmatrix} \right\}$, and an orthonormal basis for the nul space is $\left\{ \begin{bmatrix} \sqrt{\frac{1}{6}} \\ -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{6}} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{2}{15}} \\ -\sqrt{\frac{1}{30}} \\ -\sqrt{\frac{8}{15}} \\ \sqrt{\frac{3}{10}} \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{1}{10}} \\ 0 \\ -\sqrt{\frac{1}{10}} \\ -\sqrt{\frac{2}{5}} \\ \sqrt{\frac{2}{5}} \end{bmatrix} \right\}$.
(2) Diagonalize B into PDP^T then take square root for D. The result is long to write down and can be obtained via Mathematica or SageMath. We will not have computation as complicated as this in the final exam.

The following is the result from Sagemath:

$$\begin{bmatrix}
\sqrt{\frac{64\sqrt{2}+299}{1405}} & \sqrt{\frac{112\sqrt{2}+172}{1405}} & \sqrt{\frac{80\sqrt{2}+163}{1405}} & \sqrt{\frac{-32\sqrt{2}+272}{1405}} & \sqrt{\frac{-224\sqrt{2}+499}{1405}} \\
\sqrt{\frac{112\sqrt{2}+172}{1405}} & \sqrt{\frac{392\sqrt{2}+883}{2810}} & \sqrt{\frac{140\sqrt{2}+1058}{1405}} & \sqrt{\frac{-112\sqrt{2}+4043}{2810}} & 7\sqrt{\frac{-8\sqrt{2}+68}{1405}} \\
\sqrt{\frac{80\sqrt{2}+163}{1405}} & \sqrt{\frac{140\sqrt{2}+1058}{1405}} & \sqrt{\frac{100\sqrt{2}+2803}{1405}} & \sqrt{\frac{-40\sqrt{2}+5398}{1405}} & \sqrt{\frac{-280\sqrt{2}+8843}{1405}} \\
\sqrt{\frac{-32\sqrt{2}+272}{1405}} & \sqrt{\frac{-112\sqrt{2}+4043}{2810}} & \sqrt{\frac{-40\sqrt{2}+5398}{1405}} & \sqrt{\frac{32\sqrt{2}+20803}{2810}} & \sqrt{\frac{112\sqrt{2}+17032}{1405}} \\
\sqrt{\frac{-224\sqrt{2}+499}{1405}} & 7\sqrt{\frac{-8\sqrt{2}+68}{1405}} & \sqrt{\frac{-280\sqrt{2}+8843}{1405}} & \sqrt{\frac{112\sqrt{2}+17032}{1405}} & \sqrt{\frac{784\sqrt{2}+27899}{1405}}
\end{bmatrix}$$

2. Consider the matrix $\begin{bmatrix} a & 0 & b \\ 3/5 & 0 & 4/5 \\ 0 & -1 & 0 \end{bmatrix}$.
(1) Find all a, b so that is it not invertable.
(2) Find all a, b so that is it has determinant 1.
(3) Find all a, b so that it is orthogonal.

Solution: (1) $4a = 3b$. (2) $4a - 3b = 5$. (3) $a = \pm 4/5$, $b = \mp 3/5$.

3. True or false:
(1) If A and B has the same nul space then $rank(A^T B) = rank(B)$.

Solution: False. For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

(2) If A is diagonalizable, and has only two eigenspaces which are orthogonal complements of each other, then A is symmetric.
Solution: True. The union of orthonormal basis of these two subspaces is an orthonormal basis of $\mathbb{R}^n$ where A is of size $n \times n$. So, $A = PDP^T$ for some orthogonal matrix P, hence A is symmetric.