Topics:
- $L^2$ Betti numbers, Novikov-Shubin invariant, $L^2$ torsion.
- Determinant and approximation conjecture.

1 Notations and Reviews

$\mathbb{C}[G]$: group algebra of $G$. $\ell^2(G)$: $L^2$ summable functions on $G$, can be seen as a $L^2$ completion of $\mathbb{C}[G]$. $\mathcal{N}(G)$: $G$-equivariant (with left $G$ action) bounded linear maps on $\ell^2(G)$. A finite dimensional Hilbert $\mathcal{N}(G)$-module is a $G$-Hilbert space that $G$-equivariantly isometrically embedded into $\mathbb{C}^n \otimes \ell^2(G)$. Unless specified otherwise, all spaces are assumed to be finitely dimensional Hilbert $G$ modules.

1.1 Spectral theory of bounded self adjoint operators

Let $H$ be a Hilbert space. A bounded operator $A$ on $H$ is called self adjoint if $A^* = A$, positive if $(Ax, x) > 0$ for all $x \in H$.

$E$ is called a spectral measure if it sends Borel sets in $\mathbb{R}$ to bounded positive self adjoint maps on $H$, satisfying the following:

- $E(\emptyset) = 0$, $E(\mathbb{R}) = I$
- $E(A)^2 = E(A)$
- $E(A \cap B) = E(A)E(B)$
- If $A \cap B = \emptyset$, $E(A \cup B) = E(A) + E(B)$
- $\forall x, y \in H$, $(E(\cdot)x, y)$ is a $\mathbb{C}$-measure.

Theorem 1

$A$ is bounded self adjoint on $H$ then there is some spectral measure $E$ such that $A = \int AE(\lambda)$. If $f$ is an analytic function, $f(A) = \int f(\lambda)E(\lambda)$

Remark 1. When $H$ is finite dimensional, $E = \sum \delta_{\lambda_i} v_i v_i^*$.

2 $L^2$-Betti numbers

2.1 Definitions

- $G$ is a group, $f$ is a $G$-equivariant self adjoint bounded operator from $\mathbb{C}^n \otimes \ell^2(G)$. The $L^2$ trace is defined as $tr_G(f) = \sum_i (f(e_i \otimes 1), e_i \otimes 1)$. The $L^2$-trace of a self map on Hilbert $G$ modules are defined as the composition of projection and this self map.

- Let $M \subset \mathbb{C}^n \otimes \ell^2(G)$ be a Hilbert $G$ module, $pr_M$ the orthogonal projection on $M$. Then the $L^2$-dimension is defined as $\dim_G M = tr_G(pr_M)$. (Exercise: prove that the $L^2$-dimension does not depend on the choice of the embedding.)
• **$L^2$-chain complex** of finitely dimensional Hilbert $G$ modules is a sequence $\cdots \to C_{k+1} \to C_k \to C_{k-1} \to \cdots$ such that the composition of two successive boundary maps is $0$. The homology are $H^{(2)}_k = \ker(C_k \to C_{k-1})/\text{im}(C_{k+1} \to C_k)$.

• Let $X$ be a CW-complex with a free, cellular $G$ left action such that $G \setminus X$ is finite. Then $L^2$-dimension of the homology of the $L^2$ completion of the cellular chain complex $(C^{(2)}_\bullet)$ are called the $L^2$ **Betti numbers** $b^{(2)}_\bullet$.

### 2.2 Examples

**Example 2.** $X = \mathbb{R}^2$ tiled by unit cubes, $G = \mathbb{Z}^2$ (with generators $a, b$), $G \setminus X = T^2$. The $L^2$ chain complex is

$$0 \to l^2 \to (l^2)^2 \to l^2 \to 0$$

such that $\partial_2(x) = ((xa-x), (x-xb))$, $\partial_1(x, y) = xb-x+ya-y$. By computation we see $H^{(2)}_* = 0$, hence $b^{(2)}_* = 0$.

**Example 3.** $X$ a double cover of the $\theta$ graph unwrapping over one of the two loops, $G = \mathbb{Z}/2$, $G \setminus X$ is the $\theta$-shaped graph. $H^{(2)}_1$ is of dimension $3$, $(pr_{H^1_1}(e), e)$ can be computed explicitly, and $b^{(2)}_1 = 3/2, b^{(2)}_0 = 1/2$.

**Remark 4.**

- When $X$ is a finite simplicial graphs, $pr_{H^1_1}(e)$ can be calculated via spanning trees: Let $\mathcal{T}$ be the set of all spanning trees on $X$. For any $T \in \mathcal{T}$, let $\text{path}(T, e^-, e^+)$ be the path on $T$ from $e^-$ to $e^+$. Then

$$pr_{H^1_1}(e) = e - \frac{1}{|\mathcal{T}|} \sum_{T \in \mathcal{T}} \text{path}(T, e^-, e^+)$$

- There is also a physical interpretation of $pr_{H^1_1}(e)$ via electrical currents.

- In general, if $G$ is a finite group, $b^{(2)}_k(X) = b_k(X)/|G|$.

**Example 5.** $X$ is the universal cover of $\theta$-shaped graph, $G = F_2$ the deck transformation. $pr_{H^1_1}$ can be explicitly calculated (hint: first show that the element in $C^{(2)}_1$ of a complete binary tree whose root is at the root that minimizes the norm has norm $1$) $(pr_{H^1_1}(e, e)) = 1/3$, $b^{(2)}_1 = 1$, $b^{(2)}_0 = 0$.

**Remark 6.** There are alternative interpretations of the computation above through electrical currents and random walks.
2.3 Elementary properties of $L^2$ dimension, $L^2$ homology and $L^2$ Betti numbers

Some elementary properties of $L^2$ trace:

- $f \leq g \implies tr_G(f) \leq tr_G(g)$
- If $f_i$ is increasing and weakly converging to $f$, $tr_G f = \sup \{ tr_G(f_i) \}$.
- $f \geq 0$, $tr_G(f) = 0 \iff f = 0$.
- $tr_G(f + \lambda g) = tr_G(f) + \lambda tr_G(g)$
- $f$, $g$ and $h$ are self-adjoint maps compatible with an exact sequence of Hilbert $G$ modules, then $tr_G(g) = tr_G(f) + tr_G(h)$.
- $f : U \to V$, then $tr_G(f^*f) = tr_G(ff^*)$
- $f$ and $g$ are maps on Hilbert $G$ and $H$ modules, then $tr_{G \times H} f \otimes g = tr_G f \otimes tr_H g$.
- $H$ is a finite index subgroup of $G$, then $tr_H f = [G : H]tr_G f$

Proof: use definition and functional analysis.

Some elementary properties of $L^2$-dimensions:

- $dim_G(V) = 0 \iff V = 0$.
- $0 \to U \to V \to W \to 0$ weakly exact ($L^2$ homology vanishes), then $dim_G(V) = dim_G(U) + dim_G(W)$.
- $V_i$ increasing, $dim_G \bigcup_i V_i = \sup_i dim_G V_i$.
- $V_i$ decreasing, $dim_G \bigcap_i V_i = \inf_i dim_G V_i$.
- $U$, $V$ are $G$ and $H$ modules respectively, then $dim_{G \times H} U \otimes V = dim_G U \cdot dim_H V$.

Theorem 2
Let $0 \to C_* \to D_* \to E_* \to 0$ be an exact sequence of chains of $G$-modules, then there is a long exact sequence which is weakly exact.

Some elementary properties of $L^2$ Betti numbers

- $f : X \to Y$ cellular $G$-equiv maps between free $G$ cell complexes, with induced map on homology isomorphism for $p < d$ and surjective for $p = d$, then so are the induced maps on $L^2$ homology. (proof: chain homotopy, then use long exact sequence)
- $X$ free $G$ cell complex with $G \setminus X$ finite. Then $\chi(G \setminus X) = \sum_k (-1)^k b_k^{(2)}(X)$. 


• $X$ is a cocompact $d$-dimensional manifold, then $b_p^{(2)} = b_{d-p}^{(2)}$.

• Künneth formula for products, formula for wedges, connected sums for manifolds of dimension at least 3, Morse inequalities all same as the usual Betti numbers.

• If $X$ is connected, $b_0^{(2)} = 1/|G|$.

$[G:H] < \infty$, then $X$ seen as $H$ complex has $L^2$-Betti numbers $[G:H]$ of it seen as $G$ complex.

**Theorem 3**

If a cellular map of a finite connected complex, $T_f$ its mapping tori, $\pi_1(T_f) \to G \to \mathbb{Z}$ for some $G$, then the $G$-cover of $T_f$, denoted as $\overline{T_f}$ and seen as $G$-complex has zero $L^2$ Betti numbers.

**Proof.** Let $G_n$ be the preimage of $n\mathbb{Z}$ in $G \to \mathbb{Z}$. Then $\overline{T_f}$ has $n$-times as much $L^2$-Betti numbers, however $G_n \backslash T_f = T_{f,n}$ has bounded number of cells, hence all Betti number has to be 0. □

Remark 7. The rationality of $b_k^{(2)}$ for cofinite complexes are called Atiyah conjecture. It is known to be true for some classes of groups but not true in general.

### 3 Approximation for subgroups of finite index

**Theorem 4**

$X$ is a cell complex with free cellular $G$ action as before, $G\backslash X$ finite. $G \supset G_1 \ldots$ normal subgroups such that $\cap_i G_i = 1$, $[G : G_i] < \infty$, then $b_k^{(2)}(X) = \lim_{i \to \infty} b_k^{(2)}(G_i \backslash X)$, the latter as $G/G_i$ complexes.

This can be easily reduced to the following “algebraic” statement:

**Proposition 8**

Suppose $f$ is a positive self adjoint map on $\mathbb{C}^n \otimes l^2(G)$ induced by a (left) $\mathbb{Z}[G]$ module homomorphism, $f_i$ be the induced maps on $\mathbb{Z}[G/G_i]$, then $\dim_G \ker(f) = \lim_{i \to \infty} \dim_{G/G_i} \ker(f_i)$.

**Proof.** Step 1: Let $K$ be $n^2$ of the largest sum of all coeff of an entry in the matrix representing $f$, then it is larger than the operator norm of both $f$ and $f_m$.

Step 2: The map can be represented as a right multiplication of a $\mathbb{Z}[G]$-matrix, hence $tr_G(f) = tr_{G/G_i}(f_i)$ for large enough $i$. Furthermore, for any polynomial $p$, $tr_G(p(f)) = tr_{G/G_i}(p(f_i))$ for large enough $i$.

Step 3: Let $F, F_i$ be the spectral density function for $f$ and $f_i$ ($F(\lambda) = \dim_G(E([0,\lambda]))$, $F_i(\lambda) = \dim_{G/G_i}(E_i([0,\lambda]))$). Let $\overline{F}$, $\underline{F}$ be the lim sup and lim inf of $F_i$. We shall prove that $\overline{F} \leq F \leq \underline{F}^+$. Let $p_n$ be polynomials above $\chi([0,\lambda])$ and below $\chi([0,\lambda] + 1/n) + 1/n\chi([0, K])$ slightly above that. Then

$$\underline{F}(\lambda) \leq tr_G(p_n(f)) \leq F(\lambda + 1/n) + 1/n$$
And as \( n \to \infty \) the middle term converges to \( F(\lambda) \) due to spectral decomposition.

Step 4: We now prove that \( F_i \) are uniformly right-continuous at 0. This is due to a fact in linear algebra:

**Lemma 9.** \( f \): self adjoint positive linear map on \( \mathbb{C}^n \). \( K \) a bound on operator norm of \( f \), \( C \) a lower bound on the first non-zero term of characteristic polynomial of \( f \). Then for \( \lambda < 1 \),

\[
\frac{\text{num. of roots in } (0, \lambda)}{n} \leq \frac{-\log(C)}{n(-\log(\lambda))} + \frac{\log(K)}{-\log(\lambda)}
\]

**Proof.** Count non-zero roots.

Because the matrix is integral \( C \) can be chosen uniformly as 1, which finishes the proof.

**Example 10.** \( X \) is the universal cover of closed surfaces, \( G \) the deck group.

**Example 11.** Let \( \Gamma \) be a finite graph, \( \Gamma \leftarrow \Gamma_1 \leftarrow \ldots \) regular covers, for every edge \( e \in \Gamma \), let \( d_i(e) \) be the ratio of spanning trees of \( \Gamma_i \) that doesn’t contain a specific lift of \( e \). Then \( d_i \) converges.

### 4 Other \( L^2 \) invariants

\( f: U \to V \), \( F \) is the spectral density function such that \( F(\lambda) = \dim_G(\text{im}(Ef^*(\lambda,2))) \).

- **Novikov-Shubin invariants** \( \alpha(F) = \lim \inf_{\lambda \to 0^+} \frac{\log(F(\lambda) - F(0))}{\log \lambda} \)
- **Fuglede-Kadison determinant** \( \det = \exp(\int \log(\lambda) dF) \).
- **\( L^2 \) torsion** \( \rho^{(2)}(X) = -\sum_{p} (-1)^p \det(\partial_p) \). Well defined when all \( L^2 \) Betti numbers equals 0. (Motivation: finite dimensional case).

**Remark 12.**

- Determinant conjecture: For any group \( G \), any \( \mathbb{Z}[G] \) matrix \( f \), the F-K determinant of \( f^*f \) is at least 1.
- Determinant conjecture implies approximation for any sequence of subgroups.

\( X \) is a mapping torus of pseudo-Anosov with cell decomposition, choose and fix a fundamental domain, \( \rho^{(2)}(X, t, \phi) \) be the \( L^2 \) torsion twisted at some first cohomology class \( \phi \) by \( t \in (0, \infty) \). Then:

- (Lueck, Schink) \( \rho^{(2)}(X, 1, \phi) = -\frac{\text{vol}(X)}{\text{tr}}. \)
- (Liu, Friedl, Lueck) \( \lim_{t \to 0} \frac{\rho^{(2)}(X,t,\phi)}{\log(t)} - \lim_{t \to \infty} \frac{\rho^{(2)}(X,t,\phi)}{\log(t)} = \|\phi\| \).