Kazhdan's theorem for canonical metric on graphs

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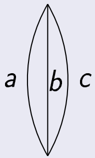
May 28, 2019

An elementary question

Let G be a finite metric graph, $l : E(G) \to \mathbb{R}^+$ the length function.
For any spanning tree T, $w(T) = \prod_{e \notin T} l(e)$
For any edge e,

$$l_{can}(e) = \frac{\sum_{e \notin T} w(T)}{\sum_T w(T)}$$

Example

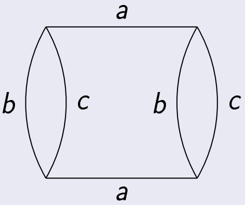


$l(a) = 2,\ l(b) = l(c) = 1.$ Then

$$l_{can}(a) = \frac{l(a)l(c) + l(a)l(b)}{l(a)l(c) + l(a)l(b) + l(b)l(c)} = \frac{4}{5}, l_{can}(b) = l_{can}(c) = \frac{3}{5}$$

Example, cont.

Consider this double cover G_1:



$l_{can}(a) = 2/5$, $l_{can}(b) = l_{can}(c) = 11/20$.

Statement of the result

Theorem [Shokrieh, W]

- G: finite metric graph
- $G \leftarrow G_1 \leftarrow G_2 \leftarrow \dots$: infinite tower of regular covers
- $\pi_i : G_i \to G$ covering maps
- $l_i : E(G) \to \mathbb{R}$, such that $\pi_i^* l_i = l_{can}$

Then l_i converges.

In the example above, if $\cap_i \pi_1(G_i) = 1$, the limit will be $(11 - \sqrt{41})/10$, $(\sqrt{41} - 1)/20$.

Interpretation of l_{can}

1. Counting spanning trees.
2. Foster's coefficient for resistor network:
   $l_{can}(e) = \frac{l(e)}{l(e)+\text{Effective resistance of } G \setminus e}$
3. Graph version of the canonical/Arakelov metric (Zhang, Baker-Farber, Chinburg-Rumely, Amini, etc):
   $\{\omega_i\}$: orthonormal basis of harmonic 1-forms

$$l(e)l_{can}(e) = \max_{\|w\| \leq 1, \omega \text{ harmonic } 1-form} |\omega(e)|^2 = \sum_i \omega_i^2(e)$$

Analogies between graphs and surfaces:

- Outer Space vs. Teichmuller space
- Berkovich space

Motivation on Riemann Surface

Theorem [Kazhdan, 70s]

- S: a compact Riemann surface
- $d^S_{can} = \sum_i |\omega_i|^2$; $\{\omega_i\}$: Orthonormal basis of $\Omega^1(S)$
- $S \leftarrow S_1 \leftarrow S_2 \leftarrow \dots$: infinite tower of regular covers, $\cap_i \pi_1(S_i) = 1$
- d_i: Riemannian metrics on S whose pull-back on S_i are the $d^{S_i}_{can}$

Then d_i converges uniformly to a multiple of the hyperbolic metric.

- "Graph version" of hyperbolicity and uniformization

Proof sketch

1. Define the limit metric.
2. Upper bound: monotonicity in collapsing subgraphs
3. Same volume: Lück's appromimation

Theorem [Lück]

X: CW complex with Γ-action which is free and cellular, X/Γ finite, $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \dots$ finite index normal subgroups of Γ, $\cap_i \Gamma_i = 1$. Then

$$\lim_{i \to \infty} \frac{b_j(X/\Gamma_i)}{[\Gamma : \Gamma_i]} = b_j^{L^2}(X)$$

Generalizations

The above-mentioned result can be generalized to the following cases with identical proofs:

- Zhang's admissible metric
- Compact Riemann surfaces and Riemannian manifolds.
- Compact flat surfaces with Delaunay triangulation.
- ...

References

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