Asymptotic translation lengths on sphere complex

Chenxi Wu

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Curve graph and Curve complex

Let S be a closed surface with genus > 1.

- Curve graph:
  - Vertices are isotopy classes of simple closed curves
  - Two vertices are connected by an edge of length 1 if the corresponding curves are disjoint.
- Curve complex:
  - Vertices are isotopy classes of simple closed curves
  - Vertices form a simplex iff the corresponding curves are disjoint.
- Curve graphs are Gromov hyperbolic. (Masur-Minsky)
- The mapping class group acts on it by isometry.
- The action is hyperbolic (has 2 fixed points at the boundary) iff the mapping class is pseudo-Anosov. (Masur-Minsky)

Asymptotic translation length on curve graphs

- Asymptotic translation length, or stable length: $l(g) := \lim_{n\to\infty} \frac{d(v, g^n v)}{n}$, where v is a vertex of the curve graph and d the distance in curve graph.
- $l(g) > 0$ iff g pseudo-Anosov.
- $l(g)$ are rational numbers, with denominator bounded by some number depending only on the genus (Bowditch). Hence $l(g)$ can be calculated by finding geodesics on the curve graphs.
- $l(g) \gtrsim g(S)^{-2}$, where $g(S)$ is the genus, and this lower bound is optimal. (Gadre-Tsai)
- $l(g) \gtrsim g(S)^{-1}$, when g is in the Torelli group, and the lower bound is optimal. (Baik-Shin)

Thurston's norm and fibered cone

- Kin-Shin: the example in Gadre-Tsai for pseudo-Anosov maps with small asymptotic translation lengths can be made to be within a single fibered cone.
- Baik-Shin-Wu: one can further find an upper bound for all maps within the same fibered cone.

- Thurston's fibered cone:
  - $\psi : S \to S$ pseudo-Anosov, $M = S \times [0,1]/\sim$, $(\psi(x), 0) \sim (x, 1)$: mapping torus of ψ. $\alpha \in H^1(M; \mathbb{Z})$ pullback from the projection on S^1.
  - Thurston norm: $\beta \in H^1(M; \mathbb{Z})$, $\|\beta\| = \min_S \max\{0, -\sum_i \chi(S)\}$ where S is a dual of β.
  - Thurston norm can be extended to $H^1(M; \mathbb{R})$ as PL norm, the unit ball is a rational polytope. The cone over the fact which contains α is the fibered face containing α, in which any primitive integer class β represents a fiber of M over the circle, hence a pseudo-Anosov map ψ_β on the fiber S_β.
- Theorem (Hyungryul Baik-Hyunshik Shin-W) Let L be a rational slice of a proper subcone of the fibered cone P, passing through origin. Then, for any primitive integer element $\beta \in L$, $l(\psi_\beta) \lesssim \|\beta\|^{-1-1/(d-1)}$, where $d = \dim(L)$.

Analogy for sphere complex

Let M be a 3-manifold. Let F_n be the free group with n generators.

- Sphere complex $S(M)$:
  - Vertices: isotopy classes of embedded spheres.
  - Faces: disjoint spheres.
- $Homeo(M)$ acts on it by isometry.
- When M is doubled handlebody, this is the free splitting complex of F_g, which is Gromov hyperbolic, and $Out(F_n)$ elements act on it loxodromically iff the train track representation has a filling lamination (which is weaker than being fully irreducible). (Handel-Mosher, Aramayona-Souto)
- Hatcher-Vogtmann also described a connection between sphere complex and free factor complex.

- Generalized fibered cone
  - In the case of surfaces, the fibered cone can be obtained as follows: let $\tilde{S}$ be the maximal free abelian cover where ψ lifts, D a fundamental domain, Γ the deck group. Then H^1 of the mapping torus is the dual of $\Gamma \otimes \mathbb{Z}$. Let Ω_i be the convex hull in $\mathbb{Z}^n$ of elements g such that $(-gD) \cap \tilde{\psi}^i(D) \neq \emptyset$. $\Omega = \cup_i \Omega_i \times \{i\}$. Then the fibered cone containing ψ is the asymptotic dual cone of Ω. (See e.g. Baik-Shin-W).
  - The same can be generalized to arbitrary dimensions.
- Theorem (Hyungryul Baik-Dongryul Kim-W) Let L be a rational slice of a proper subcone of the generalized fibered cone P, passing through origin. Then, for any primitive integer element $\beta \in L$, $l(\psi_\beta) \lesssim \|\beta\|^{-1-1/(d-1)}$, where $d = \dim(L)$, l is the asymptotic translation length on $S(M_\beta)$.
- Application to free factor and free splitting complexes via Dowdall-Kapovich-Leininger.

Proof idea for $d = 2$

M: surface or 3-manifold, N: mapping torus

- $\psi : M \to M$. Lift it to an invariant $\mathbb{Z}$ fold cover $\tilde{M} \to M$ as $\tilde{\psi}$. Let h be the deck transformation.
- Let $\tilde{N}$ be the $\mathbb{Z}^2$ cover of N, deck transformation group is generated by $\{\Psi, H\}$. Let $\{e_1, e_2\}$ be the dual basis. Then $\psi^p_{(p,q)} = \tilde{\psi}$, $S_{(p,q)} = \tilde{S}/\langle \tilde{\psi}^q h^{-p} \rangle$.
- When p is large, let D be a fundamental domain of $\tilde{M}$, then h^{-p} can be far from $\tilde{\psi}^q$, hence, if c is a curve in one fundamental domain in between, it would take many iterations of $\tilde{\psi}^q$ till it covers the whole $M_{(p,q)}$.

Remaining questions

Remaining questions:

- Is the upper bound optimal? (known in only a single example, by Eiko Kin, Baik, Shin and Wu)
- Idea: bound the minimal iterates needed for an edge in the train track representation to cover the whole graph.
- Can one get lower bound for the FF_n case?