Asymptotic translation lengths on sphere complex

Chenxi Wu

April 1, 2021
Curve graph and Curve complex

Let $S$ be a closed surface with genus $> 1$.

- **Curve graph:**
  - Vertices are isotopy classes of simple closed curves
  - Two vertices are connected by an edge of length 1 if the corresponding curves are disjoint.

- **Curve complex:**
  - Vertices are isotopy classes of simple closed curves
  - Vertices form a simplex iff the corresponding curves are disjoint.

- Curve graphs are Gromov hyperbolic. (Masur-Minsky)
- The mapping class group acts on it by isometry.
- The action is hyperbolic (has 2 fixed points at the boundary) iff the mapping class is pseudo-Anosov. (Masur-Minsky)
Asymptotic translation length on curve graphs

- Asymptotic translation length, or stable length:
  \[ l(g) := \lim_{n \to \infty} \frac{d(v, g^n v)}{n}, \]
  where \( v \) is a vertex of the curve graph and \( d \) the distance in curve graph.

- \( l(g) > 0 \) iff \( g \) pseudo-Anosov.

- \( l(g) \) are rational numbers, with denominator bounded by some number depending only on the genus (Bowditch). Hence \( l(g) \) can be calculated by finding geodesics on the curve graphs.

- \( l(g) \gtrsim g(S)^{-2} \), where \( g(S) \) is the genus, and this lower bound is optimal. (Gadre–Tsai)

- \( l(g) \gtrsim g(S)^{-1} \), when \( g \) is in the Torelli group, and the lower bound is optimal. (Baik-Shin)
Thurston’s norm and fibered cone

- **Kin-Shin**: the example in Gadre-Tsai for pseudo-Anosov maps with small asymptotic translation lengths can be made to be within a single fibered cone.
- **Baik-Shin-Wu**: one can further find an upper bound for all maps within the same fibered cone.
Thurston’s fibered cone:

- $\psi : S \rightarrow S$ pseudo-Anosov, $M = S \times [0, 1]/\sim$, $(\psi(x), 0) \sim (x, 1)$: mapping torus of $\psi$. $\alpha \in \text{H}^1(M; \mathbb{Z})$ pullback from the projection on $S^1$.

- Thurston norm: $\beta \in \text{H}^1(M; \mathbb{Z})$,
  $$\|\beta\| = \min_S \max\{0, -\sum \chi(S)\}$$
  where $S$ is a dual of $\beta$.

- Thurston norm can be extended to $\text{H}^1(M; \mathbb{R})$ as PL norm, the unit ball is a rational polytope. The cone over the fact which contains $\alpha$ is the fibered face containing $\alpha$, in which any primitive integer class $\beta$ represents a fiber of $M$ over the circle, hence a pseudo-Anosov map $\psi_\beta$ on the fiber $S_\beta$.

Theorem (Hyungryul Baik-Hyunshik Shin-W) Let $L$ be a rational slice of a proper subcone of the fibered cone $P$, passing through origin. Then, for any primitive integer element $\beta \in L$, $l(\psi_\beta) \lesssim \|\beta\|^{-1-1/(d-1)}$, where $d = \dim(L)$. 
Analogy for sphere complex

Let $M$ be a 3-manifold. Let $F_n$ be the free group with $n$ generators.

- Sphere complex $S(M)$:
  - Vertices: isotopy classes of embedded spheres.
  - Faces: disjoint spheres.

- $\text{Homeo}(M)$ acts on it by isometry.

- When $M$ is doubled handlebody, this is the free splitting complex of $F_g$, which is Gromov hyperbolic, and $\text{Out}(F_n)$ elements act on it loxodromically iff the train track representation has a filling lamination (which is weaker than being fully irreducible). (Handel-Mosher, Aramayona-Souto)

- Hatcher-Vogtmann also described a connection between sphere complex and free factor complex.
Generalized fibered cone

In the case of surfaces, the fibered cone can be obtained as follows: let \( \tilde{S} \) be the maximal free abelian cover where \( \psi \) lifts, \( D \) a fundamental domain, \( \Gamma \) the deck group. Then \( H^1 \) of the mapping torus is the dual of \( \Gamma \otimes \mathbb{Z} \). Let \( \Omega_i \) be the convex hull in \( \mathbb{Z}^n \) of elements \( g \) such that \( (-gD) \cap \tilde{\psi}^i(D) \neq \emptyset \). \( \Omega = \cup_i \Omega_i \times \{i\} \). Then the fibered cone containing \( \psi \) is the asymptotic dual cone of \( \Omega \). (See e.g. Baik-Shin-W).

The same can be generalized to arbitrary dimensions.

Theorem (Hyungryul Baik-Dongryul Kim-W) Let \( L \) be a rational slice of a proper subcone of the generalized fibered cone \( P \), passing through origin. Then, for any primitive integer element \( \beta \in L \), \( l(\psi_\beta) \lesssim \|\beta\|^{-1-1/(d-1)} \), where \( d = \dim(L) \), \( l \) is the asymptotic translation length on \( S(M_\beta) \).

Application to free factor and free splitting complexes via Dowdall-Kapovich-Leininger.
Proof idea for $d = 2$

$M$: surface or 3-manifold, $N$: mapping torus

- $\psi : M \to M$. Lift it to an invariant $\mathbb{Z}$ fold cover $\tilde{M} \to M$ as $\tilde{\psi}$. Let $h$ be the deck transformation.

- Let $\tilde{N}$ be the $\mathbb{Z}^2$ cover of $N$, deck transformation group is generated by $\{\Psi, H\}$. Let $\{e_1, e_2\}$ be the dual basis. Then $\psi^p_{(p,q)} = \tilde{\psi}$, $S_{(p,q)} = \tilde{S}/\langle \tilde{\psi}^qh^{-p} \rangle$.

- When $p$ is large, let $D$ be a fundamental domain of $\tilde{M}$, then $h^{-p}$ can be far from $\tilde{\psi}^q$, hence, if $c$ is a curve in one fundamental domain in between, it would take many iterations of $\tilde{\psi}^q$ till it covers the whole $M_{(p,q)}$. 
Remaining questions

- Is the upper bound optimal? (known in only a single example, by Eiko Kin, Baik, Shin and Wu)
- Idea: bound the minimal iterates needed for an edge in the train track representation to cover the whole graph.
- Can one get lower bound for the $FF_n$ case?