1. A first example

1.1. **From infinite translation surface map to end-periodic map.** We begin with an infinite half-translation surface M^0_∞ described as in Figure 1 and an affine map f_0 defined as follows: the surface is horizontally stretched by $\frac{\sqrt{5}-1}{2}$ and vertically shrinked by $\frac{2}{\sqrt{5}-1}$, then the part to the left of the vertical dashed line is sent to the part above the horizontal dashed line, while the part to the right of the vertical dashed line is rotated by π and sent to the part below the horizontal dashed line.

Let M^1_∞ be the double cover of M^0_∞ which is an infinite translation surface. More precisely, consider two copies of the polygon in figure 1: P_0 and P_1. For each pair of edges in figure 1 with the same label, if they are in the same direction, we glue the pairs in P_0 and P_1 together; if they are in the different direction, we glue the edges in P_0 to the edges in P_1 in the opposite direction. Let f_1 be the lift of f_0 that sends the part of P_0 to the left of the vertical dashed line to the part of P_0 above the horizontal dashed line of the same leaf.

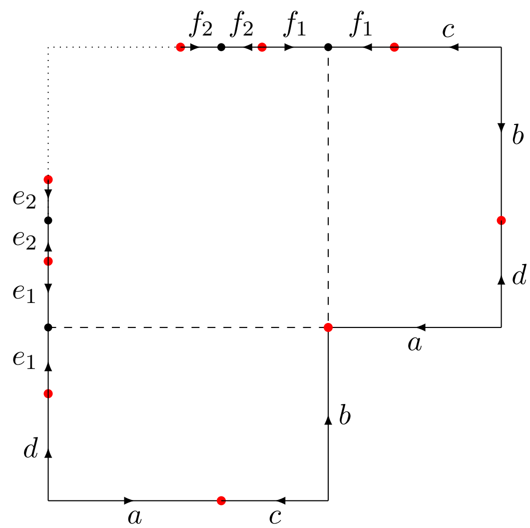

Figure 1

Now we “blow up” M^1_∞ into M^3_∞ as shown in figure 2. The “blow up” proceeds as follows: start with the polygonal region in figure 1. Replace each red dot in the boundary with a short arc g_n, and add a line at the top-left corner which is divided into short segments h_n. Glue all the other edges according to the previous paragraph, then we get an infinite (topological) surface M^2_∞ with boundary whose boundary curves are g_n and h_n. f_1 lifts to f_2, which sends g_n to g_{n+1} and h_n to h_{n+1}. Keep the gluing relations on all other sides. Let M^3_∞ be the double of M^2_∞ along its boundary, and f_3 be a diffeomorphism from M^3_∞ to itself which is identical to f_2 when double cover of f_2. Now f_3 is an end-periodic map (c.f. [F97, CC]). This "blow up" process reverses the Handel-Miller construction [CC], where pair of invariant laminations, hence an affine map on half-translation surface, is obtained from an end-periodic map.

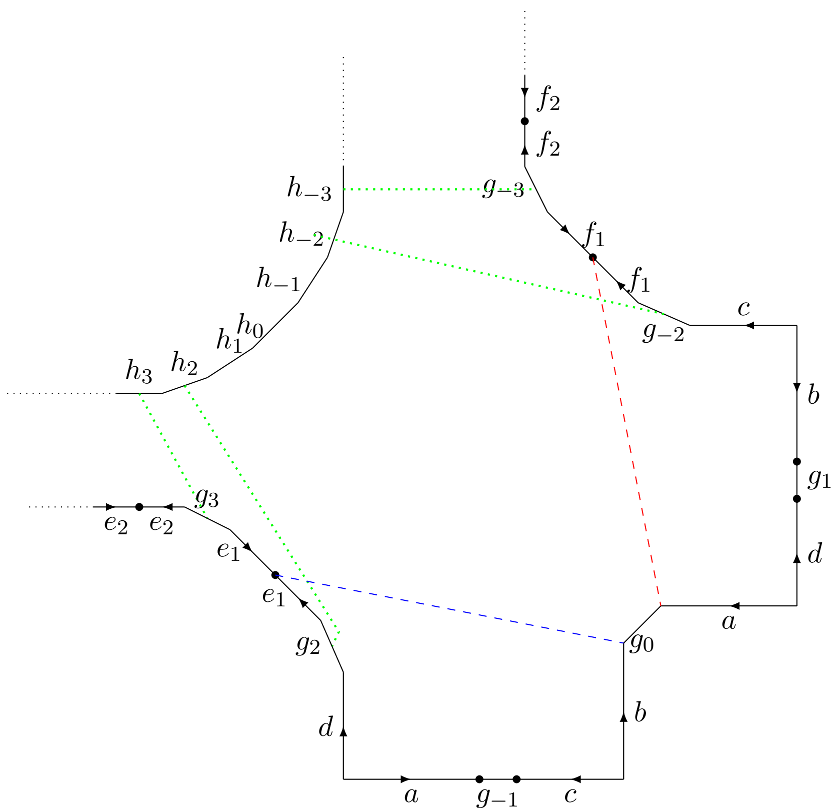

FIGURE 2. This is one of the two leaves of X_∞. g_i and h_i are the boundary curves. S_∞ is its double.

The map $\tilde{F}$ is defined as sending the region to the left of the red line to the region above the blue line, and sending the region to the right of the red line to the part below the blue line on another leaf.

1.2. **From end-periodic map to 3-manifold.** Let T be the mapping torus of M^3_∞ with monodromy f_3. This is a non compact 3-manifold, since it is a mapping torus of a non compact surface. The mapping torus construction restricted to the two periodic ends (the attracting one and the repelling one) forms two cylindrical ends of the 3 manifold as in Figure 3, also c.f. [F97]. Hence, it has a natural compactification $\overline{T}$ by adding two copies of a closed surface homeomorphic to the quotient of either end of M^3_∞ by f_3. More precisely, in T, consider the part coming from the suspension of a neighborhood of the repelling end, and adjoin an ideal point to every flow line in there. The set of such ideal points is homeomorphic to the quotient of this neighborhood under the map f_3. Similarly, one can do the same thing with the attracting end. In particular, the boundary curves of these two periodic neighborhoods (in our case, the green dotted lines in figure 2) are identified with union of loops C and C' on these two boundary surfaces.

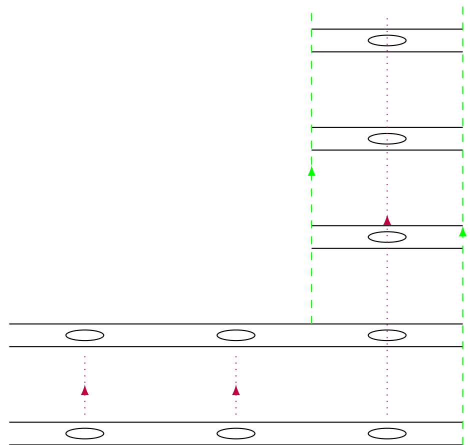

FIGURE 3

1.3. **Approximation by a sequence of finite type surfaces.** Glue the two boundary surfaces of $\overline{T}$ together such that C and C' are identified, then we get a closed 3-manifold with a depth 1 foliation, whose infinite leaves are from the mapping torus and the compact leaf is from the boundary surface of $\overline{T}$, whose dual in $H^1(N)$ is denoted as b. Remove a small cylindrical neighborhood of the compact leaf and glue the two ends of the infinite leaves togather, we get a fiberation of N on a circle which corresponds to a point close to b in $\mathbb{P}H^1(N)$.

In terms of the surface, what this gives us is a map f_4 on a finite surface M^4 based on the end-periodic map. M^4 is formed by gluing the preimage of the green dotted loops under some iterate of f_3 in the attracting end to the green dotted loops in the repelling end, and f_4 is the same as f_3 except on the last "step" in the attracting end, which, instead of being sent to the next step, it is sent to the step in the repelling end glued to it. The exact way it is sent to the repelling end is determined by how we identify the boundaries of $\overline{T}$ to form N. Here, we send h_n to h_{-3}, g_n to g_{-3}, and e_{n-1} to f_1 as shown in figure 2. Varying n we get a sequence of maps on surfaces of increasing genus, which represents an arithmetic sequence in a fibered cone of N.

We can give a flat structure on each finite surface map by invariant measure laminations, making them two copies of the double cover of half translation surfaces. In our case, one of them is a cover of the following:

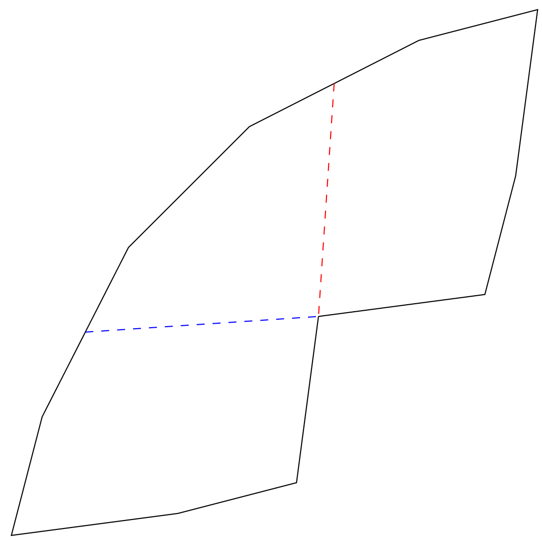

FIGURE 4. The dilatation constant λ satisfies $\lambda^6 - \lambda^4 - \lambda^3 - \lambda^2 + 1 = 0$

And, by calculation, we know that the dilatation of f_4 on these surfaces satisfy $\lambda^n - \lambda^{n-2} - \cdots - \lambda^2 + 1 = 0$.

1.4. **Fibered face and Teichmüller polynomial.** The 1-eigenspace of ${f_4}^*$ on $H^1(M^4;\mathbb{Q})$ has rank 3 and is generated by the duals of the two loops formed by h_i as well as the dual of the green loops as shown in figure 2. Hence, the compact 3-manifold has first Betti number 4. The loops formed by h_i are both preserved by f_4, hence perturbing a fiberation in those two directions correspond to doing Dehn twists on embedded tori, which changes neither genus (i.e. they lie in the kernel of the Thurston norm) nor the dilatation constant when those loops are collapsed.

2. Other examples

Example 1. Bowman [JB] described such a sequence formed by Arnoux-Yoccoz surfaces.

Example 2 (Chamanara surface[RC]). Start with the Chamanara surface (Figure 5) with the Baker's map, where the part to the right of the vertical dashed line is sent to the part below the horizontal dashed line:

In the polygon in Figure 5, replace each blue dot in the boundary with a short arc, and add a line at the top-left corner and a line at the bottom-right

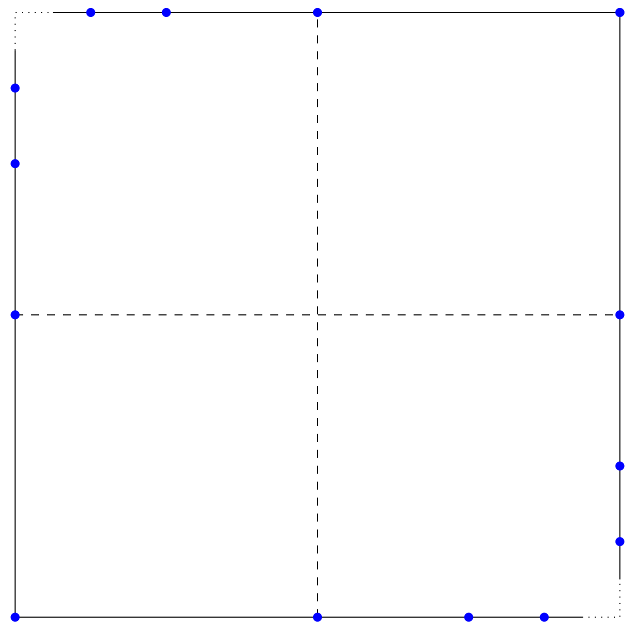

FIGURE 5

corner, which are both divided into short segments. Glue all the other edges according to the previous paragraph, then we get an infinite (topological) surface with boundary and the affine map lifts to an end-periodic map with one contracting and one repelling end. Do a double of this infinite surface along the boundary, and follow the same procedure as described in the previous section, we get a sequence of finite surface maps approximating the Baker's map on the Chamanara surface.

Figure 6 is a surface in the sequence approximating the Chamanara surface.

It's in strata $\mathcal{H}(2)$. The dilatation constant is approximately 1.7221 which is the root of $x^4 - x^3 - x^2 - x + 1 = 0$.

The next surface in this sequence is Figure 7.

It's in strata $\mathcal{H}(1,1)$. The dilatation constant is approximately 1.8832 which is the root of $x^5 - x^4 - x^3 - x^2 - x + 1 = 0$. In general, the dilatation constant is a root of $x^n - x^{n-1} - \cdots - x + 1 = 0$.

Example 3. Consider a piecewise linear map from a square to itself defined as follows: decomposes the square horizontally into three rectangles of the same width, stretch them horizontally by 3 and shrink them vertically by 3, then stack them vertically such that the left-most rectangle is put in

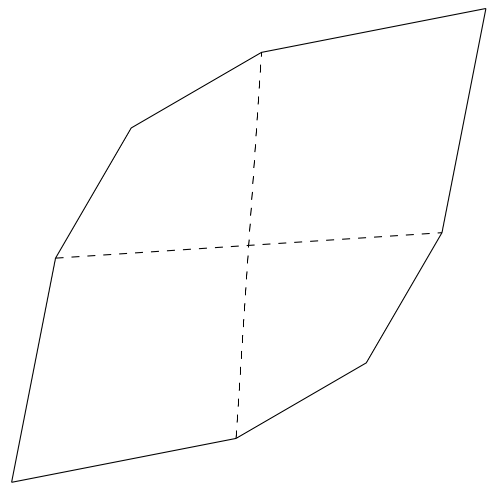

Figure 6

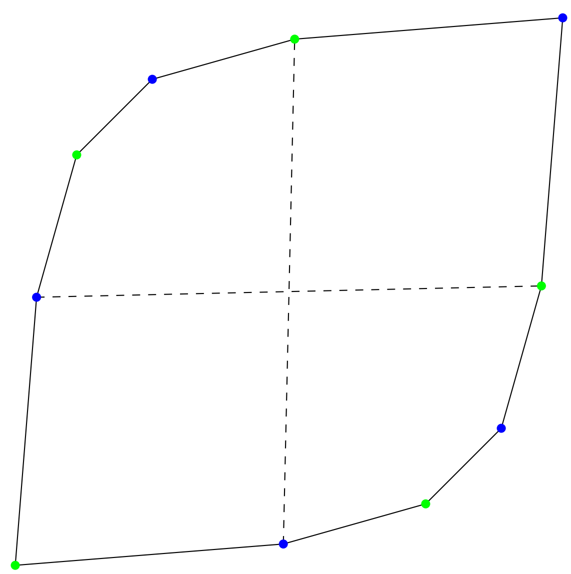

Figure 7

the middle, the middle rectangle is put at the bottom, and the right-most rectangle is rotated by π and put at the top. This map induces a gluing on the boundary of the square which makes it into a half-translation surface of infinite type, as shown in Figure 8.

Pass to a double cover that makes it a translation surface, then replace the infinite cone points with line segments and do a double along them as in the previous section, we get a end-periodic map on a infinite-type surface with one contracting and one repelling end. Following the same process as

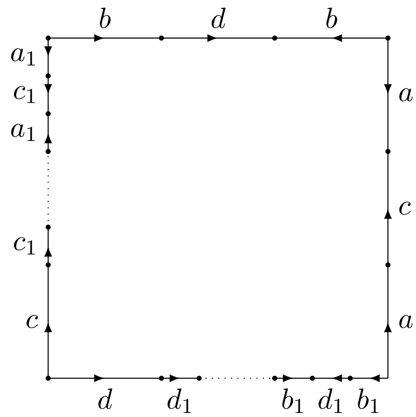

FIGURE 8

described in the previous section we can get a sequence of finite surface with pseudo-Anosov maps, one of which is shown in Figure 9.

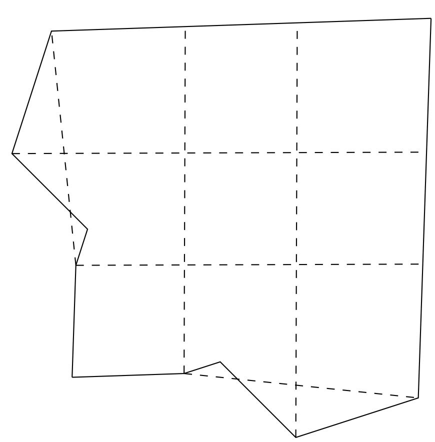

FIGURE 9. The dilatation of this pseudo-Anosov map is the largest root of $\lambda^8 - 3\lambda^7 - 2\lambda^5 + 4\lambda^4 - 2\lambda^3 - 3\lambda + 1 = 0$.

By calculation, the dilatation of these sequence of pseudo-Anosov mapd satisfies $\lambda^{8+2n} - 3\lambda^{7+2n} - 2\lambda^{5+n} + 4\lambda^{4+n} - 2\lambda^{3+n} - 3\lambda + 1 = 0$.

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