Feb 10/11

1. True or false: if A is a $n \times n$ matrix, $A\mathbf{x} = 0$ has non-zero solution if and only if there are some $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has no solution.

Answer: This is true. Firstly we prove the "if" part: let $\mathbf{b}$ be a vector such that $A\mathbf{x} = \mathbf{b}$ has no solution. Consider the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$, do row reductions to make it into reduced echelon form, then there must be a pivot in the last column. Because there can only be at most n pivots, there must be an i, $1 \leq i \leq n$, such that the i-th column does not contain a pivot. Now let $\begin{bmatrix} A & 0 \end{bmatrix}$ goes through the same sequence of row reductions, we get a reduced echelon matrix with no pivots in the i-th column and with the last column 0. Hence $A\mathbf{x} = 0$ has infinitely many solutions. In particular, it has a non-zero solution.

Now we show the "only if" part. If $A\mathbf{x} = 0$ has a non-zero solution, it has infinitely many solutions. Hence, after we make $\begin{bmatrix} A & 0 \end{bmatrix}$ into a reduced echelon form through a sequence of row reductions, it can have at most $n-1$ pivots. Hence, the last row of this reduced echelon matrix is 0. Now replace the zero on the n-th row and $n+1$-th column with 1, and reverse the sequence of row reductions, we get a matrix of the form $\begin{bmatrix} A & \mathbf{b_0} \end{bmatrix}$. As row reductions of augmented matrix do not change the solution set, $A\mathbf{x} = b_0$ has no solution.

2. True or false: A and B are $n \times n$ matrices, $Ax = b$ and $By = c$ are both consistent, then $\begin{bmatrix} A & B \end{bmatrix} z = b + c$ is consistent.

Answer: This is true. Suppose $Ax_0 = b$, $By_0 = c$, the rule of matrix multiplying with a vector implies that $\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = Ax_0 + by_0 = b + c$. Hence, $z = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is a solution of $\begin{bmatrix} A & B \end{bmatrix} z = b + c$.

3. $v_1 = (1, 1, 0)$, $v_2 = (0, 1, 2)$, find $v_3 = (a_1, a_2, a_3)$ such that $x_1 v_1 + x_2 v_2 + x_3 v_3 = (1, 1, 1)$ has a solution.

Answer: The augmented matrix of the given linear system is

$$\begin{bmatrix} 1 & 0 & a_1 & 1 \\ 1 & 1 & a_2 & 1 \\ 0 & 2 & a_3 & 1 \end{bmatrix}$$

.

With row reductions, we can make it into the following form:

$$\begin{bmatrix} 1 & 0 & a_1 & 1 \\ 0 & 1 & a_2 - a_1 & 0 \\ 0 & 0 & 2a_1 - 2a_2 + a_3 & 1 \end{bmatrix}$$

.

Hence, it is consistent if and only if $2a_1 - 2a_2 + a_3 \neq 0$.

4. Find A and $\mathbf{b}$ such that the solution of $A\mathbf{x} = \mathbf{b}$ is a line passing through $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Answer: The line passing through $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ can we written as $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$, where $t \in \mathbb{R}$. In other words, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we have $x_1 = 1 - t$, $x_2 = t$, $x_3 = 2t$. Solve t from any of the three equations and substitute the value of t to the other two, we get a linear system with the required solution set. For example, we can substitute t with x_2 in the first and the third equation and get:

$$\begin{cases} x_1 = 1 - x_2 \\ x_3 = 2x_2 \end{cases}$$

Hence, we can choose $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

It is obvious that the solution of this problem is not unique. You may get some other A and $\mathbf{b}$ with a different approach.

Feb 24/25 Prelim 1 Review

- Vectors and matrices
  - Vectors and vector arithmetics
  - linear combination, span, and linear dependence
  - Matrix-vector product (as linear combination of column vectors, with row-column rule)
- Linear systems
  - Augmented matrix and coefficient matrix
  - Row reduction
  - echelon form, pivots
  - parametric vector form
  - vector equation and matrix equation
- Linear transformation
  - The definition of linear transformation

- Linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ can be written as matrix transformation i.e. multiplication by a matrix.
- The matrices of some transformations on $\mathbb{R}^2$

Last Quiz: Echelon form of $[a_1 \quad a_2 \quad a_3]$, where a_1 and a_2 are linearly independent and a_3 is in the span of $\{a_1, a_2\}$.

Because linearly independence and dependence do not change under row reduction, the first two columns must be pivot columns and the third must not be pivot column, hence pivots positions are $(1,1)$ and $(2,2)$.

Practice problems:

True or False:
1. The sum of two linear transformations is linear.
True
2. The product of a linear function and a linear transformations is linear.
False
****
3. a is in $span\{b, c\}$, then b is in $span\{a, c\}$.
False, for example if $a = 2c$, and b and c are linearly independent.
4. $[a \quad b \quad c]$ and $[a \quad c \quad b]$ has the same pivot positions.
False. For example, if $a = (1,1)$, $b = (2,2)$, $c = (1,2)$.
****
5. A is a 3×3 matrix with 2 pivots, then the solution of $Ax = b$ can never span $\mathbb{R}^3$.
True. The solution of $Ax = b$ is either empty or a line, hence can not contain 3 linearly independent vectors.

Find the matrix representing a reflection over $y = 2x$.

The vector $(1,0)$ is sent to $(\cos(\theta), \sin(\theta))$, where $\theta = 2\arctan(2)$. Hence, by trigonometry, $\cos(\theta) = -0.6$, $\sin(\theta) = 0.8$. The image of $(0,1)$ is the image of $(1,0)$ rotated clockwise by $\pi/2$, hence must be $(0.8, 0.6)$. As a result, the matrix is $\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}$.

$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & t & 0 \\ 0 & -1 & t \end{bmatrix}$. Find all t such that $x \mapsto Ax$ is not one-to-one.

Do row reduction we get:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & t \\ 0 & 0 & t^2 - 1 \end{bmatrix}$$

The map is not one-to-one if and only if A has a column without a pivot if an only if $t^2 - 1 = 0$, i.e. $t = \pm 1$.

Mar 2/3

Quiz:

Suppose A, B and X are $n \times n$ matrices, A, X and $A - AX$ are all invertible, and $(A - AX)^{-1} = X^{-1}B$. Solve for X (i.e. write X as a formula involving A and B). If you need to invert a matrix, justify that it is invertible.

Solution: $(A - AX)X^{-1}B = I$ so $AX^{-1}B - AB = I$, $AX^{-1} - A = B^{-1}$, $X = (B^{-1} + A)^{-1}A$. B is invertible because $B = X(A - AX)^{-1}$ which is a product of two invertible matrices. $B^{-1} + A$ is invertible because $B^{-1} + A = AX^{-1}$, the right-hand-side is a a product of two invertible matrices.

- Matrix multiplication ($A[b_1 \quad b_2 \dots] = [Ab_1 \quad Ab_2 \dots]$), transpose, inverse ($A^{-1}A = AA^{-1} = I$)
- Inverse formula for 2×2 matrix
- Algorithm for inverse with row reduction, which can be justified by multiplication with elementary matrices or by simultaneously solving a sequence of linear systems.
- As a consequence of the above, A invertible $\iff$ #row=#column=#pivot $\iff$ $x \mapsto Ax$ is one-to-one and onto
- vector space & subspace
- null space=kernel=solution set of $Ax = 0$; column space=image= span of columns

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Practice problems (Not part of the quiz!):

(1) Find the inverse of $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: use the row reduction algorithm. The answer is $\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$.

(2) True or false: A is invertible iff A^T is invertible.

This is true. $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$, $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$.

(3) True or false: The set of $n \times n$-invertible matrices a subspace of $M_{n \times n}$.

False. 0 is not invertible.

(4) True or false: Let V be a vector space, T be a linear map from V to V, then the intersection of the kernel (null space) and image (column space) of T is 0.

False. For example, $V = \mathbb{R}^2$, $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then the null space and column space of T are both spanned by e_1.

(5) True or false: A is a $n \times n$ matrix. If $A^3 = 0$ then $I - A$ is invertible.

True. $(I - A)(I + A + A^2) = I^3 - A^3 = I$.

Mar 9/10

Quiz: Assume that $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Find basis for Nul A and Col A.

Answer: The first, third and fifth columns are pivot columns, so a basis for Col A is $\{(1, 2, 1, 3), (-5 - 5, 0, -5), (-3, 2, 5, -2)\}$.

The solution of $Ax = 0$ is $\{t_1(-2, 1, 0, 0, 0) + t_2(-1/4, 0, 7/5, 1, 0)\}$. So a basis would be $\{(-2, 1, 0, 0, 0), (-1/4, 0, 7/5, 1, 0)\}$.

The solution is not unique, but you must justify it.

(i) Basis: (1) linearly independent (2) spans the whole space. If a finite set satisfies (2), a basis can be obtained by removing redundant elements, or see below.

(ii) Find basis of null space: solve LS. Find basis of col space: pivots.

(iii) Change-of-coordinate: if the basis of a new coordinate system, written under the current one, is $b_1, \ldots b_n$, then left-multiplying $[b_1 \quad b_2 \ldots b_n]^{-1}$ changes old coordinate to the new.

(iv) Axioms of vector space: (1) Sum exists. (2) Commutativity for addition. (3) Association for addition. (4) 0 exists. (5) Negativity exists. (Remark: (1)-(5) means that a vector space is an abelian group for addition.)
(6) Scaler multiplication exists. (7)-(8) distribution. (9) association for multiplication. (10) 1.

(v) A vector space must contain 0 hence non-empty.

(vi) The addition and scaler multiplication may not be "obvious": $\mathbb{R}^+$ is a vector space with multiplication and power.

(vii) Ways to show a subset is a vector space: show that it contains 0 and is closed under addition and multiplication, show that it is spanned by some vectors, show that it is the null/col space of a map

(viii) Remember to check "for every".

(ix) Kernel=Nul, range=image=Col

Practice problems (Not part of the quiz!):

(1) Let L be the subspace of $\mathbb{R}^4$ spanned by $(1,-1,0,0)$, $(0,1,-1,0)$, and $(-1,0,1,0)$. Find a basis for L.

Write a matrix with the 3 vectors as columns, then by row reduction one can see that the first two columns are pivot columns. Hence $\{(1,-1,0,0),(0,1,-1,0)\}$ forms a basis.

(2) Consider the map $F : M_{2\times 2} \to \mathbb{R}^2$, $F(M) = M\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) Show that F is linear.

$$F(A+B) = (A+B)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} + B\begin{bmatrix} 0 \\ 1 \end{bmatrix} = F(A)+F(B),$$
$$F(cA) = (cA)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c(A\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = cF(A).$$

(b) Use the basis $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$, write down the matrix of F.

Evaluate the basis vectors with F, one gets $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) What are the kernel and range of F?

The columns of the matrix in (b) spans $\mathbb{R}^2$, so the range of F is $\mathbb{R}^2$. Also from the matrix one can see that the kernel of F is spanned by the first and the third basis elements.

(3) True or false: $\mathbb{R}^3$ has infinitely many basis.

True. For example, all triples of the form $\{(0,0,1),(0,1,0),(1,t,0)\}$ forms a basis.

(4) True or false: if A is an invertible matrix, the columns of B is a basis of Col B, then the columns of BA is a basis of Col B.

True. The columns of BA spans Col BA, which is $Range(BA)$. Since A is onto, $Range(BA) = Range(B) =$ Col B (Because given any two mapd f and g, if g is onto then the range of $f \circ g$ is always the same as the range of f) So the columns of BA spans Col B.

The columns of B is a basis implies that they are linearly independent, hence B is one-to-one. Since A is invertible it is also one-to-one, so BA, being the composition of two one-to-one maps, is one-to-one, hence the columns of BA are also linearly independent.

Now we know that the columns of BA are linearly independent and spans Col B, so it is a basis of Col B.

Remark: Hence, b, b' is a basis implies $b + b', b - b'$ is a basis.

(5) Find a linear map whose kernel is the subspace L in problem (1).

$L = span\{(1, -1, 0, 0), (0, 1, -1, 0)\} = \{t_1(1, -1, 0, 0) + t_2(0, 1, -1, 0)\} = \{(t_1, -t_1 + t_2, -t_2, 0)\}$, so:

$$\begin{cases} x_1 = t_1 \\ x_2 = -t_1 + t_2 \\ x_3 = -t_2 \\ x_4 = 0 \end{cases}$$

Eliminate t_1 and t_2 from the above equations we have

$$\begin{cases} x_2 + x_1 + x_3 = 0 \\ x_4 = 0 \end{cases}$$

, so the map

$$x \mapsto \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$

has kernel L. The solution is obviously not unique.

(6) Let $C([0, 1])$ be the vector space of continuous functions on $[0, 1]$. Which of the followings are subspaces of L?

(a) The set of polynomials with integer coefficients.

No. Not closed under scaler multiplication.

(b) $\{f \in C([0,1]) : f(\frac{1}{3}) = 0\}$.

Yes. It is the kernel of $f \mapsto f(1/3)$.

(c) $\{f : \int_0^1 f(x)dx = 1\}$.

No. Does not contain 0.

(d) $\{f : f \text{ is differentiable}\}$.

Yes. Because of the linearity of derivatives.

(e) $\{f : f \geq 0\}$.
No. Not closed under negation or scaler multiplication.

(7) $P_3 = \{c_0 + c_1 x + c_2 x^2 + c_3 x^3\}$ is the vector space of degree 3 polynomials.

(a) Show that $T : P_3 \to P_3$, $T(f) = (xf)' - 3f$ is a linear map.

$T(cf) = (xcf)' - 3cf = c((xf)' - 3f) = cT(f)$, $T(f+g) = (x(f+g))' - 3(f+g) = (xf)' - 3f + (xg)' - 3g = T(f) + T(g)$.

(b) Write down a basis for the kernel and range of T.

$T(c_0 + c_1 x + c_2 x^2 + c_3 x^3) = (c_0 x + c_1 x^2 + c_2 x^3 + c_3 x^4)' - 3(c_0 + c_1 x + c_2 x^2 + c_3 x^3) = c_0 + 2c_1 x + 3c_2 x^2 + 4c_3 x^3 - 3(c_0 + c_1 x + c_2 x^2 + c_3 x^3) = -2c_0 - c_1 x + c_3 x^3$. So a basis of the range of T is $\{1, x, x^3\}$, and a basis of its kernel is $\{x^2\}$.

Mar 16/17

Quiz: $\mathcal{B} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis of $\mathbb{P}_2$. Find the coordinate vector of $p = 3 + t - 6t^2$ under $\mathcal{B}$.

Review:

(i) Change of coordinate: start with a basis $\mathcal{B}_0$ (e.g. standard basis for $\mathbb{R}^n$, or $1, t, t^2, \ldots$ under $\mathbb{P}_n$). For any basis $\mathcal{B}$, let P be matrix whose columns are the coordinates of $\mathcal{B}$ under $\mathcal{B}_0$. We have:

- To get coordinate under $\mathcal{B}_0$ from coordinate under $\mathcal{B}$, left multiply with P.
- To get coordinate under $\mathcal{B}$ from coordinate under $\mathcal{B}_0$, left multiply with P^{-1}.
- To get coordinate under $\mathcal{C}$ from coordinate under $\mathcal{B}$, do the above 2 successively.

(ii) $\dim V =$ size of a basis of V
$=$ size of any basis of V
$=$ maximum number of linearly independent elements in V
$=$ minimal number of elements that span V

(iii) A is $m \times n$, then $rank A =$ dim Col A
$=$ dim Row A
$= n -$ dim Nul A
(Row $A =$ Col A^T).

(iv) Basis theorem: if $\dim V = n$, n elements in V span V iff they are linearly independent iff they form a basis.

(v) Same dimension does not mean identical!

(vi) Row operation preserves Row, Nul but changes Col.

Practice problems (Not part of the quiz!):

(1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. What is the rank of A? Find a basis for Row, Col and Nul of A.

Answer: The rank is 2 because there are 2 pivots. By observation, one can see that a basis of Col $A =$ Row A^T is $\{(1,0,0),(3,5,0)\}$, and a basis of Row $A =$ Col A^T is $\{(1,2,3,4),(0,0,5,6)\}$.

To find Nul A, we solve $Ax = 0$ by row reduction. The reduced echelon form is $\begin{bmatrix} 1 & 2 & 0 & 2/5 \\ 0 & 0 & 1 & 6/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so a basis for Nul A is $\{(-2,1,0,0),(-2/5,0,-6/5,1)\}$.

To find Nul A^T, we solve $A^T = 0$ by row reduction. The reduced echelon form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so a basis of Nul A^T is $\{(0,0,1)\}$.

(2) True or false: If A is $m \times n$ with linearly independent rows, then the dimension of Nul A is $m - n$.

Answer: False. Because all rows are linearly independent, $rank(A) = m$. dim Null $A = n - rank(A) = n - m$ which is generally different from $m - n$. For example, when $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

(3) A and B are $n \times n$ matrices, and $AB = 0$, what is the maximum of dim Col $A +$ dim Row B?

Answer: $AB = 0$ means that Col B must be contained in Nul A. (Col $B =$ Range $(x \mapsto Bx) = \{Bx\}$, and any element of the form Bx must be in Nul A because $A(Bx) = (AB)x = 0x = 0$.) Hence, *dim* Row $B =$ dim Col $B \leq$ dim Nul $A = n - rank(A) = n -$ dim Col A, so dim Col $A +$ dim Row $B \leq n$. The bound n is also the maximum, because when $A = 0$, $B = I$, this sum is n.

Remark: for the version on your handout (find the maximum of dim Nul $A +$ dim Row B), the trivial bound $2n$ is also the maximum because one can let $A = 0$ and $B = I$.

(4) What is the dimension of $\{(a+2b+c, 3a+b+3c, -a-c, a+b+c)\}$?

Answer: The space is spanned by $\{(1,3,-1,1),(2,1,0,1)\}$, hence has dimension 2.

(5) What is the coordinate of $x^3 \in \mathbb{P}_3$ under $\{1, x-1, (x-1)^2, (x-1)^3\}$?

Answer: You can use change-of-coordinate formula, or do it directly with binomial formula: $x^3 = ((x-1)+1)^3 = (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$.

(6) True or false: $rank(AB) \leq \min\{rank(A), rank(B)\}$.

Answer: True. $rank(AB)$ is the dimension of the image of $x \mapsto ABx$, which is a subspace of the image of $x \mapsto Ax$, hence $rank(AB) \leq rank(A)$. On the other hand, $rank(AB) = rank((AB)^T) = rank(B^T A^T)$, hence the same argument will show that $rank(AB) \leq rank(B^T) = rank(B)$.

Remark: $rank(AB) = rank(B) - \dim($ Col $B \cap$ Nul $A) = rank(A) - \dim($ Row $A \cap$ Nul $B^T)$.

Mar 23/24

Quiz

Find the particular solution for the difference equation $y_{k+2} - 4y_{k+1} + 3y_k = 0$, $k = 0, 1, \dots$, with boundary conditions $y_0 = 1$ and $y_N = 0$.

Answer: The auxiliary equation is $r^2 - 4r + 3 = 0$, hence the general solution is $y_k = C_1 \cdot 3^k + C_2$. Use $y_0 = C_1 + C_2 = 1$ and $y_N = C_1 \cdot 3^N + C_2 = 0$ we can get $C_1 = -\frac{1}{3^N - 1}$, $C_2 = \frac{3^N}{3^N - 1}$, $y_k = -\frac{1}{3^N - 1} 3^k + \frac{3^N}{3^N - 1}$.

Review

(i) Linear difference equation

(a) The concept of order of a difference equation.

(b) The solution set of an order k homogeneous equation has dimension k.

(c) **Show solutions linearly independent: Casorati matrix.**

(d) The solution set of a non-homogeneous equation is a particular solution plus the solution set of homogeneous equation.

(e) **Find solution for homogeneous equation by solving the auxiliary equation: if it has distinct roots $r_1, \dots r_k$, then general solution is $c_1 r_1^n + \dots c_k r_k^n$. $e^{it} = \cos t + i \sin t$.**

(f) Relation with first order systems of equation $\mathbf{x_{n+1}} = A\mathbf{x_n}$. The roots of the auxiliary equation are the eigenvalues of A.

(ii) Determinant

(a) Effect under row and column operation.

(b) Multilinearity

(c) $\det(\mathbf{I}) = \mathbf{1}$
The above 3 characterize $\det$

(d) Geometric meaning: volume.

(e) $\det(AB) = \det(A)\det(B)$

(f) $\det(A) = \det(A^T)$

Practice problems

(1) Show that the general solution of $y_{k+2} + 2y_{k+1} + y_k = 0$ is $y_k = C_1(-1)^k + C_2 k(-1)^k$.

Answer: As it is a homogeneous linear equation of order 2, its solution set is a vector space of dimension 2. Hence, by basis theorem, we only need to verify that:

(a) $(-1)^k$ and $k(-1)^k$ are both solutions: this is because $(-1)^{k+2}+2(-1)^{k+1}+(-1)^k=(-1)^k(1-2+1)=0$, and $(k+2)(-1)^{k+2}+2(k+1)(-1)^{k+1}+k(-1)^k=(-1)^k(k+2-2(k+1)+k)=0$.

(b) $(-1)^k$ and $k(-1)^k$ are linearly independent: this is because $\begin{bmatrix} (-1)^k & k(-1)^k \\ (-1)^{k+1} & (k+1)(-1)^{k+1} \end{bmatrix}$ is invertible.

(2) $\{u,v,w\}$ is a basis of $\mathbb{R}^3$, what is the relationship between $\det([u+v \quad v+w \quad w+u])$ and $\det([u \quad v \quad w])$?

Answer: $\det([u+v \quad v+w \quad w+u]) = 2\det([u \quad v \quad w])$. There are many ways of showing it:

- We can use matrix multiplication:

$$\det([u+v \quad v+w \quad w+u]) = \det\left([u \quad v \quad w]\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right)$$

$$= \det([u \quad v \quad w])\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) = \det([u \quad v \quad w])\cdot 2$$

- We can use multi-linearity of det:

$$\det([u+v \quad v+w \quad w+u]) = \det([u \quad v+w \quad w+u]) + \det([v \quad v+w \quad w+u])$$

$$= \det([u \quad v \quad w+u]) + \det([u \quad w \quad w+u]) + \det([v \quad v \quad w+u]) + \det([v \quad w \quad w+u])$$

$$= \det([u \quad v \quad w]) + \det([u \quad v \quad u]) + \det([u \quad w \quad w]) + \det([u \quad w \quad u])$$

$$+ \det([v \quad v \quad w]) + \det([v \quad v \quad u]) + \det([v \quad w \quad w]) + \det([v \quad w \quad u])$$

$$= \det([u \quad v \quad w]) + 0+0+0+0+0+0 + \det([v \quad w \quad u]) = 2\det([u \quad v \quad w])$$

(3) True or false: $\det(A+B) = \det(A)+\det(B)$.

Answer: False. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(4) True or false: If A is 4×3, then $\det(AA^T) = 0$.

Answer: True. By the rule of matrix multiplication, the columns of AA^T are linear combinations of the columns of A, hence Col $AA^T \subseteq$ Col A, hence $rank(AA^T) = \dim$ Col $AA^T \leq \dim$ Col $A \leq 3$. However, AA^T is a 4×4 matrix, hence it can never be invertible. As a consequence, $\det(AA^T) = 0$.

(5) Find all solutions of $y_{k+2} - y_{k+1} + y_k = 0$, $y_0 = y_N = 0$.

Answer: The roots of the Auxiliary polynomial $r^2 - r + 1 = 0$ are $r = \frac{1}{2} \pm \frac{\sqrt{3}}{2} = e^{\pm \frac{\pi i}{3}} z$, so the general solution is $y_k = C_1 \sin(k\pi/3) + C_2 \cos(k\pi/3)$. Use the boundary condition, we have: $y_0 = C_2 = 0$, $y_N = C_1 \sin(k\pi/3) + C_2 \cos(k\pi/3) = 0$. Hence, if N is not a multiple of 3, the only solution is $y_k = 0$; if N is a multiple of 3, the solutions are sequences of the form $y_k = C \sin(k\pi/3)$.

(6) A is a 2×2 matrix with all entries in $[-1, 1]$. What is the largest possible $\det(A)$? How about when A is 4×4?

Answer: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc \leq |a||b| + |c||d| \leq 1 + 1 = 2$, and $\det\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right) = 2$, so the largest possible $\det(A)$ is 2.

We can think about it in another way: the column vectors has length at most $\sqrt{1^2 + 1^2} = \sqrt{2}$, and the det is maximized if they can be made both orthogonal and of the greatest possible length, hence det is bounded above by $(\sqrt{2})^2 = 2$. Similarly, in 4×4 case, the column vectors are of length at most $\sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$, hence det is bounded above by $2^4 = 16$. After a few tries it is possible to find a matrix whose determinant reaches this upper bound, for instance the following one:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

Apr 6/7

Quiz

Find k in matrix $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & k & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, such that the eigenspace of A for $\lambda = 5$ is two-dimensional.

Answer: The eigenspace for 5 is the null space of $A - 5I$. It has dimension 2 if and only if the matrix $A - 5I$ has rank 2. By row operation one can see that it happens if and only if $k = 6$.

Review

- Motivation: to study linear maps from a vector space to itself, e.g. the behavior of iterations of such maps, we need to understand the normal form of a matrix under **similarity transformation** $A \mapsto P^{-1}AP$.
- **Characteristic polynomial**: $\det(A - \lambda I)$. Its roots $\lambda_1, \dots \lambda_k$ are called **eigenvalues**. When λ_j is an eigenvalue, $\ker(A - \lambda_j I)$ is called the eigenspace of λ_j. An element of an eigenspace is called an **eigenvector**.
- Characteristic polynomials, eigenvalues, dimension of the eigenspaces are all invariant under similarity.
- Eigenvectors of different eigenvalues are linearly independent.
- Diagonalizable (under similarity) iff eigenspaces span the whole vector space.

Practice problems

(1) Find the eigenvalues and eigenspaces of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Is it diagonalizable?

Answer: Denote the matrix as A, the characteristic polynomial is $\det(A - \lambda I) = -\lambda^3 + 2\lambda^2 + \lambda - 2 = -(\lambda - 2)(\lambda + 1)(\lambda - 1)$, hence the eigenvalues are 2, 1 and -1. Their corresponding eigenspaces are the null space of $A - 2I$, $A - I$ and $A + I$, respectively. By solving homogeneous linear system we know the eigenspaces for 2 is $span\{(1, 1, 1)\}$, the eigenspace for 1 is $span\{(0, 1, -1)\}$, and the eigenspace for -1 is $span\{(-2, 1, 1)\}$. The eigenvectors $(1, 1, 1)$, $(0, 1, -1)$ and $(-2, 1, 1)$ span $\mathbb{R}^3$, hence A is diagonalizable.

(2) Find a such that $\begin{bmatrix} 0 & a \\ 1 & 1 \end{bmatrix}$ is not diagonalizable in $\mathbb{R}$.

Answer: The characteristic polynomial is $\lambda^2 - \lambda - a$. When $a > -\frac{1}{4}$, this polynomial has two distinct real roots, hence the matrix has at least two linearly independent eigenvectors, hence must be diagonalizable. When $a < -\frac{1}{4}$, the characteristic polynomial has no real roots, hence the matrix has no real eigenvalue, hence must not be diagonalizable. When $a = -\frac{1}{4}$, the matrix has an unique eigenvalue $\frac{1}{2}$ with eigenspace $span\{(-1, 2)\}$, which does not span $\mathbb{R}^2$, hence is not diagonalizable.

Remark: When $a < -\frac{1}{4}$ the matrix is not diagonalizable under $\mathbb{R}$ but is diagonalizable under $\mathbb{C}$.

(3) True or false: If $A^2 = I$, any eigenvalue of A must be either 1 or -1.

Answer: True. If λ is an eigenvalue of A, there is a non-zero vector x such that $Ax = \lambda x$. Hence $x = AAx = A(\lambda x) = \lambda(Ax) = \lambda^2 x$, which implies $\lambda^2 = 1$, i.e. $\lambda = \pm 1$.

Remark: It can be further shown that the A here is diagonalizable.

(4) True or false: A is diagonalizable if and only if A^2 is diagonalizable.

Answer: False. If A is diagonalizable, i.e. $A = P^{-1}DP$ where D is diagonal, then $A^2 = P^{-1}DPP^{-1}DP = P^{-1}D^2P$ is diagonalizable. However, A^2 being diagonalizable does not imply that A is diagonalizable, for example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(5) A is an invertible square matrix with characteristic polynomial $f(\lambda)$. What is the characteristic polynomial of A^{-1}? Write it in terms of f.

Answer: Suppose A is $n \times n$, the characteristic polynomial of A^{-1} is $\det(A^{-1} - \lambda I) = \det((-\lambda)A^{-1}(A - \frac{1}{\lambda}I)) = \det(-\lambda I) \cdot \det(A^{-1}) \cdot \det(A - \frac{1}{\lambda}I) = (-1)^n \lambda^n \cdot \frac{1}{\det(A)} f(\frac{1}{\lambda}) = \frac{(-1)^n \lambda^n f(\frac{1}{\lambda})}{f(0)}$.

(6) True or false: If Nul $A \cap$ Col A has dimension 1, then A can not be diagonalizable.

Answer: True. Suppose an $n \times n$ matrix A is diagonalizable, with eigenvalues $\lambda_1, \ldots \lambda_k$, we will show that Nul $A \cap$ Col A always has dimension 0.

Because A is diagonalizable, non-zero $x \in \mathbb{R}^n$ can be uniquely written as $x = x_1 + \ldots x_k$, such that $Ax_j = \lambda_j x_j$. If none of the λ_j is 0, Nul $A = 0$, hence Nul $A \cap$ Col A has dimension 0, a contradiction. Now suppose $\lambda_1 = 0$. Then, $x \in$ Nul A iff $x = x_1$, while $x \in$ Col A iff $x_1 = 0$. Hence Nul $A \cap$ Col A has dimension 0.

(7) True or false: If λ is an eigenvalue of A, then $\lambda^3 - 1$ is an eigenvalue of $A^3 - I$.

Answer: True. λ is an eigenvalue of A means that there is a non-zero x such that $Ax = \lambda x$, hence $(A^3 - I)x = A^3 x - x = \lambda^3 x - x = (\lambda^3 - 1)x$, hence $\lambda^3 - 1$ is an eigenvalue of $A^3 - I$.

(8) Calculate $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^N$.

See Example 2 on Page 284 of the textbook.

(9) True or false: 1/2 can not be the eigenvalue of an integer matrix (a matrix whose entries are integers).

Answer: True. By definition of determinant, the characteristic polynomial of an integer matrix is a polynomial with integer coefficients and leading coefficient ± 1. Hence $\frac{1}{2}$ can not be an eigenvalue because any rational root of a monic integer polynomial is an integer. (This is more a number theory question than a linear algebra question, so don't worry about it if you find it a bit confusing.)

April 13

Quiz

Diagonalize the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ (i.e. find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$), or show that A is not diagonalizable.

Review

Matrix of a map $T : V \to W$ under base change: $\mathcal{B}, \mathcal{C}$ are bases of V, $\mathcal{D}, \mathcal{E}$ are bases of W, M is the matrix of T under $\mathcal{B}$ and $\mathcal{D}$, then the matrix of T under $\mathcal{C}$ and $\mathcal{E}$ is $P_{\mathcal{E} \leftarrow \mathcal{D}} M P_{\mathcal{B} \leftarrow \mathcal{C}}$.

Matrix of a self map under base change.

Complex eigenvalue: 2×2 case: if the eigenvector of $a - bi$ is $u + vi$, then $A = [u \quad v] \begin{bmatrix} a & -b \\ b & a \end{bmatrix} [u \quad v]^{-1}$.

Geometric meaning: rotation and scaling.

Application: Discrete dynamical system $x_{k+1} = Ax_k$. When x_k are probability vectors and A is a stochastic matrix ($[1 \ldots 1]A = [1 \ldots 1]$ and all entries of A are non-negative), it is a discrete Markov chain.

When A is diagonalizable on $\mathbb{C}$, 0 is an attractor, repeller or saddle iff all eigenvalues have magnitude < 1, > 1, or some < 1 while some > 1. Direction of greatest attraction/repulsion is the eigenspace of the smallest/largest eigenvalue.

For Markov chain, if A^k is positive for some k (called **regular**), A has a unique positive eigenvalue whose eigenvector is the steady state.

Practice Problems

1. $A = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$.

   (i) What are the complex eigenvalues and eigenvectors of A?

   (ii) Find a basis of $\mathbb{R}^2$ under which $x \mapsto Ax$ is the composition of a scaling and a rotation.

   (iii) What are the scaling and the rotation in (ii)?

   (iv) $x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x_{k+1} = Ax_k$, describe $\{x_k\}$.

Answer: (i) The eigenvalues are $1 \pm i$ and the corresponding eigenvectors are $(1, -1 \mp i)$.

(ii) From (i), we know that $x \mapsto Ax$ has matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ under $\{(1,-1),(0,1)\}$.

(iii) Scaling by $\sqrt{2}$ and rotation by $\pi/4$.

(iv) x_k spirals away towards infinity, and the direction rotates with a period of 8.

2. True or false:

(i) The product of two stochastic matrices is a stochastic matrix.

(ii) The transpose of a stochastic matrix is a stochastic matrix.

(iii) 2 can not be an eigenvalue of a stochastic matrix.

(iv) If A is a real 4×4 matrix, $2 - i$, $1 + \frac{1}{2}i$ are both eigenvalues, then 0 is a saddle for $x_{k+1} = Ax_k$

(v) If A is a stochastic matrix, then 0 is not a repeller in $x_{k+1} = Ax_k$.

Answer: (i) True. If A and B are both stochastic, all entries of A and B are non-negative, hence all entries of their product would also be non-negative. Furthermore, if $[1 \dots 1]A = [1 \dots 1]$ and $[1 \dots 1]B = [1 \dots 1]$, $[1 \dots 1]AB = [1 \dots 1]B = [1 \dots 1]$. Hence AB is also stochastic.

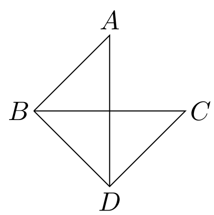
(ii) False. For example $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

(iii) False. Suppose otherwise, let A be a stochastic matrix and a_{ij} be the entry on the i-th row and j-th column, $x = (x_1, \dots x_n)$ be an eigenvector of 2, then $2\sum_i |x_i| = \sum_i |2x_i| = \sum_i |\sum_j a_{ij}x_j| \leq \sum_i \sum_j a_{ij}|x_j| = \sum_j (\sum_i a_{ij})|x_j| = \sum_j |x_j|$. Hence $\sum_i |x_i| = 0$, $x = 0$, a contradiction.

(iv) False. Because A is real, $2 + i$ and $1 - \frac{1}{2}i$ are also eigenvalues, hence all eigenvalues of A have magnitude greater than 1, which means that 0 can not be a saddle.

(v) True. Because A always has eigenvalue 1.

3. Consider a random walk on the following graph:



such that if one is at vertex v at time t, one will be at one of the vertices neighboring v with the same probability.

(i) Write down the corresponding stochastic matrix P.

(ii) Is P regular?

(iii) Find the steady-state vector for P.

Answer: (i)$P = \begin{bmatrix} 0 & 1/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 1/3 \\ 0 & 1/3 & 0 & 1/3 \\ 1/2 & 1/3 & 1/2 & 0 \end{bmatrix}$.

(ii)Yes. All entries of P^2 are positive.

(iii)In the steady state, the probabilities at A and C are 0.2 and the probabilities at B and D are 0.3.

4. Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. $T : \mathbb{P}_2 \to \mathbb{P}_2$ sends f to f'.

(i) Write down the matrix of T under basis $\{1, x, x^2\}$.

(ii) Write down the matrix of T under basis $\{1 + x^2, x + 1, x^2\}$.

Answer: To find the matrix of T under a basis $\mathcal{B} = \{b_1, \dots\}$, we put the $\mathcal{B}-$coordinate of $T(b_k)$ at the k-th column.

(i) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

(ii) $\begin{bmatrix} -2 & 1 & -2 \\ 2 & 0 & 2 \\ 2 & -1 & 2 \end{bmatrix}$.

Prelim II review

- **Matrix Arithmetics:** The definition of matrix addition, scaler multiplication, matrix multiplication (row/column reductions can be written as matrix multiplication), transpose and inverse.
  Algorithm to calculate the inverse of a matrix. How to check if a matrix is invertible.

  Exercise: If the second row of A is twice the first row of A, what is the relationship between the first two row of AB? Can AB be invertible?

  Answer: The rows of A are the columns of A^T, and the rows of AB are the columns of $(AB)^T = B^T A^T$, which, by definition of matrix multiplication, are the columns of A^T multiply with B^T from the left. Hence, the second row of AB is twice the first row of AB, and AB can not be invertible because its rows are not linearly independent.

- **Determinants:** Can be computed by cofactor expansion (p. 168). $\det(A) \neq 0$ iff A is invertible. $\det(A^T) = \det(A)$, $\det(AB) = \det(A)\det(B)$. How det changes under row/column operations. Geometric meaning as the "volumn".

  Exercise: A is an $n \times n$ invertible matrix. What is $\det(-A^T A^{-1})$?

  Answer: $\det(-A^T A^{-1}) = (-1)^n \det(A^T)\det(A^{-1}) = (-1)^n \det(A)/\det(A) = (-1)^n$.

- **Vector space and linear maps**

  - **Vector space**: a set with two operations: addition and scalar multiplication, that satisfies a sequence of axioms. $\mathbb{R}^n$ are vector spaces.
  - **Subspace**: A subset of a vector space that contains 0 and is closed under both addition and scalar multiplication. When V is a vector space, $S \subset V$, $span\{S\}$ is a subspace. The **kernel (null space)** and **range (column space)** of any linear map are subspaces of its domain and codomain, respectively. Use row reduction to compute null space and column space.
  - **Basis**: A linear independent subset that spans the vector space. Given vector space V, any two basis has the same number of elements, and the number of elements in any basis of V is called its **dimension**. The Basis theorem. The dimension of the range of a linear map is called its **rank**.
  - **Coordinates and change of basis**: Given a vector space V with basis $\mathcal{B}$, any element $v \in V$ can be written as a linear combination

of elements in $\mathcal{B}$, and the weights are its **$\mathcal{B}$-coordinates** $[v]_\mathcal{B}$. Given linear map $T : V \to W$, basis $\mathcal{B} = \{b_1, \ldots b_n\}$ in V and $\mathcal{C}$ in W, let $M = [[T(b_1)]_\mathcal{C} \ldots [T(b_n)]_\mathcal{C}]$, then $[T(v)]_\mathcal{C} = M[v]_\mathcal{B}$. How are $[]_\mathcal{B}$ and the matrix M changes when we change the basis?

Exercise: Show that $V = \{a \sin(x + b)\}$ is a subspace in the vector space of continuous functions, find a basis, write down the map $T_a : f(x) \mapsto f(x + a) - f(x)$ under this basis and calculate its rank.

Answer: We can show that $V = span\{\sin(x), \cos(x)\}$: $a \sin(x + b) = a \cos(b) \sin(x) + a \sin(b) \cos(x)$, hence $V \subseteq span\{\sin(x), \cos(x)\}$. $c \sin(x) + c' \cos(x) = \sqrt{c^2 + c'^2} \sin(x + d)$, where d is the angle between the positive x-axis and the vector (c, c'), hence $span\{\sin(x), \cos(x)\} \subseteq V$. From this we can conclude that V is a subspace of the space of continuous functions. Furthermore, $\sin(x)$ and $\cos(x)$ are linearly independent, hence they form a basis of V, and $\dim(V) = 2$. $T_a(\sin(x)) = \sin(x + a) - \sin(x) = (\cos(a) - 1) \sin(x) + \sin(a) \cos(x)$, $T_a(\cos(x)) = \cos(x + a) - \cos(x) = (\cos(a) - 1) \cos(x) - \sin(a) \sin(x)$. Hence, the matrix of T under basis $\{\sin(x), \cos(x)\}$ is

$$\begin{bmatrix} \cos(a) - 1 & \sin(a) \\ -\sin(a) & \cos(a) - 1 \end{bmatrix}$$

You may choose another basis, which will result in another matrix. By calculation, we know that it has rank 0 if $a = 2k\pi$, $k \in \mathbb{Z}$ and rank 2 if otherwise.

- **Eigenvalues and eigenvectors** Change of basis for self maps are **similarity transformations**. **Characteristic equation, eigenvalues, eigenspaces and eigenvectors**. Criteria for diagonalization. How to diagonalize. Deal with complex eigenvalues.

  Exercise: $v, w \in \mathbb{R}^n$, $v^T w \neq 0$. Show that vw^T is diagonalizable.

  Answer: $vw^T v = (v^T w)v$, hence $v^T w$ is an eigenvalue. The rank of vw^T is 1, hence 0 is an eigenvalue, and the eigenspace for 0 has dimension $n - 1$. Hence the eigenspaces of $v^T w$ span $\mathbb{R}^n$, which implies that it is diagonalizable.

- **Applications**
  - **Linear difference equation** Auxiliary equation. General solution. Boundary value problem.
  - **Discrete dynamical system** General solution when the matrix is diagonalizable. Describe of the solution based on eigenvalues and eigenspaces.

– **Markov chain** Probability vector and stochastic matrix. Existence and uniqueness of steady-state vector when the matrix is **regular**.

Exercise: Find the direction of greatest attraction and repulsion for system $x_{k+1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1/2 \end{bmatrix} x_k$.

Answer: The eigenvalues are 2, $1 \pm i$ and $-1/2$. 2 is the eigenvalue with the greatest magnitude, hence its eigenspace, $span\{(1,0,0,0)\}$, is the direction of the greatest repulsion. $-1/2$ is the eigenvalue with the greatest magnitude, hence its eigenspace, $span\{(0,0,0,1)\}$, is the direction of the greatest attraction.

April 20/21

Practice problems:

1. $V = span\{1, x^{-1}, x^{-2}\}$. $T : V \to \mathbb{R}^2$ is defined as $T(f) = \begin{bmatrix} f(1) \\ f(-1) \end{bmatrix}$. What is the matrix of T under basis $\{1, x^{-1}, x^{-2}\}$? What is the matrix of T under basis $\{1, \frac{1+x}{x}, \frac{1+x}{x^2}\}$?

Answer: To find the matrix of T, evaluate it on the basis vectors and use them as columns.

The matrix of T under $\{1, x^{-1}, x^{-2}\}$ is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, and the matrix of T under basis $\{1, \frac{1+x}{x}, \frac{1+x}{x^2}\}$ is $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 0 \end{bmatrix}$.

2. If A is a 3×3 real matrix, both 0 and $1+i$ are eigenvalues of A. What are the (real and complex) eigenvalues of $(A+I)^{-1}$? What is the rank of A?

Answer: Because A is real and $1+i$ is an eigenvalue, its conjugate $1-i$ must also be an eigenvalue. Because A is 3×3 and has 3 distinct eigenvalues, their eigenspaces must all have dimension 1, hence the null space of A, a.k.a. the 0-eigenspace of A, must have dimension 1, hence the rank of A is $3-1=2$.

If $Ax = \lambda x$ then $(A+I)x = (1+\lambda)x$, hence the eigenvalues of $A+I$ are $1, 2\pm i$, therefore $A+I$ is invertible. As a consequence, $(1+\lambda)^{-1}x = (A+I)^{-1}x$, hence the eigenvalues of $(A+I)^{-1}$ are 1, $\frac{2\pm i}{5}$.

3. You are walking on an infinite plane, start by facing north. After each step, you turn left by $\pi/2$ with a probability of 0.2. What is the probability of you facing east after N steps as $N \to \infty$?

Answer: Let the probability of facing north, west, south and east at step k be p_k^1, p_k^2, p_k^3 and p_k^4 respectively, let $p_k = (p_k^1, p_k^2, p_k^3, p_k^4)$, then $p_{k+1} = Pp_k$, where $P = \begin{bmatrix} 0.8 & 0 & 0 & 0.2 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{bmatrix}$. Hence finding steady-state probability vector is solving $(P-I)p = 0$ and normalize your result into a probability vector. The answer is $p = (1/4, 1/4, 1/4, 1/4)$, hence the probability of facing east as $N \to \infty$ converges to 1/4.

4. Diagonalize $M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$.

Answer: The rank of M is 1, hence its null space (i.e. eigenspace for 0) has dimension 2. By solving $Ax = 0$ we know that it has a basis $\{(1,-1,0),(0,1,-1)\}$. By observation or computation, $(0,1,2)$ is an eigenvector of M with corresponding eigenvalue 3. Hence $M = PDP^{-1}$, where $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$.

5. A is a 3×3 matrix. $A^2 = A$. Use $x = Ax + (I - A)x$ to show that the eigenspaces of 0 and 1 of A span $\mathbb{R}^3$, and conclude that A is diagonalizable.

Answer: Let $V_0 = \{x : Ax = 0\}$, $V_1 = \{x : Ax = x\}$. Because $A(Ax) = A^2x = Ax$, $Ax \in V_1$. Because $A((I-A)x) = (A-A^2)x = 0x = 0$, $(I-A)x \in V_0$, hence $x = Ax + (I - A)x$ implies that V_0 and V_1 span $\mathbb{R}^3$. In other words, if $V_0 = 0$, then $V_1 = \mathbb{R}^3$, i.e. the 1-eigenspace of A is $\mathbb{R}^3$. If $V_1 = 0$, then V_0, which is the 0-eigenspace, is $\mathbb{R}^3$. If neither V_0 nor V_1 is 0, then V_0 is the 0-eigenspace and V_1 is the 1-eigenspace and together they span $\mathbb{R}^3$. In all cases $\mathbb{R}^3$ is spanned by a collection of eigenvectors of A, hence A is diagonalizable.

6. $x_n \in \mathbb{R}^2$ for all $n \in \mathbb{Z}$, $x_{n+2} = 3x_{n+1} + \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} x_n$. Find general solution.

Answer: Let $x_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$, then the linear difference equation on x can be reduced to $a_{n+2} = 3a_{n+1} - 2a_n$ and $b_{n+2} = 3b_{n+1}$, hence the general solution is $a_n = C_1 + C_2 2^n$, and $b_n = C_3 3^n$.

Alternatively, you can use the idea on p.252 of the textbook and reduce it to discrete dynamical system:

$$y_n = \begin{bmatrix} a_n \\ b_n \\ a_{n+1} \\ b_{n+1} \end{bmatrix}$$

$$y_{n+1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} y_n$$

And find general solution by diagonalizing the 4×4 matrix.

April 27/28

Quiz

Find the general solution of $x' = Ax$, $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$.

Answer: The eigenvalues of A are $2 \pm i$ and the corresponding eigenvectors are $(-1, 1 \mp i)$. Hence the general solution is:

$$Ae^{t(2+i)}(-1, 1-i) + Be^{t(2-i)}(-1, 1+i)$$

Review

(i) Differential equations: **decouple by diagonalization**, deal with complex eigenvalues, attractor, repeller, saddle and spiral point.
2×2 with complex eigenvalue $a+bi$ with corresponding eigenvector $u+vi$: $C_1(u \cos bt - v \sin bt)e^{at} + C_2(u \sin bt + v \cos bt)e^{at}$.

(ii) **Inner product in $\mathbb{R}^n$ ($x \cdot y = x^T y$).** Basic properties. **Length, distance and angles.** Orthogonal vectors. Orthogonal complement of a subspace of $\mathbb{R}^n$.

(iii) **Orthogonal set and orthogonal basis.** Orthogonal sets of non-zero vectors are linearly independent. Calculate coefficient under orthogonal basis by inner product (geometric interpretation: orthogonal projection to a line). **Orthonormal set and orthonormal basis.**

(iv) Matrix with orthonormal columns. $U^T U = I, ||Ux|| = ||x||, (Ux) \cdot (Uy) = x \cdot y$. Orthogonal matrix.

(v) **Orthogonal projection to a subspace. Calculation using an orthogonal (and orthonormal) basis. Best approximation.**

Practice problems

(1) Find an orthonormal basis of the space spanned by $(1, 0, -1)$ and $(0, 1, -1)$.

Answer: Use Gram-Schmidt. Let $v_1 = (1, 0, -1)$, $v_2 = (0, 1, -1)$, then we first replace v_2 with $v_2' = v_2 - \frac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1 = v_2 - \frac{1}{2} v_1 = (-1/2, 1, -1/2)$. Normalize them, we have $\{\frac{1}{\sqrt{2}}(1, 0, -1), \sqrt{\frac{2}{3}}(-1/2, 1, -1/2)\}$.

(2) Let $a = (0, 1, 2)$, $b = (1, 1, 1)$.

(a) Find x, y such that $c = (1, x, y)$ is orthogonal to $W = span\{a, b\}$.

(b) Let P_M be the orthogonal projection to W. Find the general solution of $x' = P_M x$.

(c) How about $x' = P_{M^\perp}x$?

Answer:

(a) $a \cdot c = 0$ means $x + 2y = 0$, $b \cdot c = 0$ means $1 + x + y = 0$. Hence $x = -2$, $y = 1$.

(b) By the definition of orthogonal projection, $P_M a = a$, $P_M b = b$, $P_M c = 0$. So a, b and c are eigenvectors of P_M and they span $\mathbb{R}^3$, hence the general solution is

$$x(t) = C_1 e^t(0,1,2) + C_2 e^t(1,1,1) + C_3(1,-2,1)$$

(c) By the definition of orthogonal projection (or because $P_{M^\perp} = I - P_M$), $P_{M^\perp}a = P_{M^\perp}b = 0$, $P_{M^\perp}c = c$. Hence the general solution is

$$x(t) = C_1(0,1,2) + C_2(1,1,1) + C_3 e^t(1,-2,1)$$

(3) True or false:

(a) $(a \cdot b)c = (b \cdot c)a$.

(b) U and V are subspaces, $W = U \cap V$, P_U, P_V and P_W are orthogonal projections to U, V and W. Then $P_W = P_U P_V$.

(c) If A is an orthogonal matrix, λ is a real eigenvalue of A. Then $|\lambda| = 1$.

Answer:

(a) False. For example, let $a = (1,0)$, $b = c = (0,1)$.

(b) False. For example, let $U = span\{(1,0,0),(0,1,0)\}$ and $V = span\{(1,0,0),(0,1,1)\}$.

(c) True. Because if U is orthogonal, $||Ux|| = ||x||$. Let x be an eigenvector of λ, then $||Ux|| = |\lambda|||x|| = ||x||$, hence $|\lambda| = 1$.

(4) a, b, c are three non-zero vectors in $\mathbb{R}^3$. a and b are orthogonal, the angle between c and a is $\pi/3$, the angle between c and b is $\pi/3$, what is the angle between c and its orthogonal projection to $span\{a,b\}$?

Answer: Because the angle between a and c is $\pi/3$, $a \cdot c = ||a||||c|| \cos(\pi/3) = \frac{1}{2}||a||||c||$. Similarly, $b \cdot c = ||b||||c|| \cos(\pi/3) = \frac{1}{2}||b||||c||$.

Because $\{a,b\}$ is an orthogonal basis of $span\{a,b\}$, the projection of c on $span\{a,b\}$, which we denote as c_1, is:

$$\begin{aligned} c_1 &= \frac{a \cdot c}{a \cdot a}a + \frac{b \cdot c}{b \cdot b}b \\ &= \frac{||c||}{2||a||}a + \frac{||c||}{2||b||}b \end{aligned}$$

Hence $||c_1|| = \sqrt{c_1 \cdot c_1} = \frac{\sqrt{2}}{2}||c||$, $c \cdot c_1 = \frac{||c||}{2||a||}\frac{||a||||c||}{2} + \frac{||c||}{2||b||}\frac{||b||||c||}{2} = \frac{||c||^2}{2}$, and the angle between c and c_1 is $\arccos \frac{c \cdot c_1}{||c||||c_1||} = \arccos \frac{\sqrt{2}}{2} = \pi/4$.

(5) In $\mathbb{R}^n$, find v that minimizes $2||v||^2 + ||v-a||^2 + ||v-b||^2$.

Answer: $2||v||^2+||v-a||^2+||v-b||^2 = v\cdot v+(v-a)\cdot(v-a)+(v-b)\cdot(v-b) = 4v\cdot v - 2(a+b)\cdot v + a\cdot a + b\cdot b = 4(v - \frac{a+b}{4}) - 4(\frac{a+b}{4}\cdot\frac{a+b}{4}) + a\cdot a + b\cdot b$. So the value is minimized when $v = \frac{a+b}{4}$.

(6) $a, b \in \mathbb{R}^2$, show that $(\det[a \quad b])^2 = (a\cdot a)(b\cdot b) - (a\cdot b)^2$.

Answer: $(\det[a \quad b])^2 = \det[a \quad b]\det[a \quad b]^T = \det\begin{bmatrix} a\cdot a & a\cdot b \\ b\cdot a & b\cdot b \end{bmatrix} = (a\cdot a)(b\cdot b) - (a\cdot b)^2$.

May 4/5

Quiz

A subspace W of $\mathbb{R}^4$ has basis $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix} \right\}$. Find an orthogonal basis for W.

Answer: See HW solution.

Bonus problem: $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x) = (a \cdot x)b + (b \cdot x)a$. What is the matrix of T? When is this matrix orthogonal? When is it diagonalizable (over $\mathbb{R}$)? When is it invertible? (The grade for this problem will be added to your earlier quiz grades.)

Answer: $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2b_1 \\ b_2 \end{bmatrix}$, $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} b_2 \\ 0 \end{bmatrix}$. Hence the matrix is $\begin{bmatrix} 2b_1 & b_2 \\ b_2 & 0 \end{bmatrix}$.

It is orthogonal iff the column vectors are orthogonal and of unit length, i.e. iff $2b_1b_2 = 0$ and $b_2^2 = 4b_1^2 + b_2^2 = 1$. Hence this matrix is orthogonal iff $b_2 = \pm 1$ and $b_1 = 0$.

It is invertible iff its determinant, which is $-b_2^2$, is non-zero, i.e. iff $b_2 \neq 0$.

The characteristic polynomial of this matrix is $\lambda^2 - 2b_1\lambda - b_2^2$, which has discriminant $4b_1^2 + 4b_2^2$. Hence, unless $b_1 = b_2 = 0$ it will have two different real roots hence the matrix will be diagonalizable. if $b_1 = b_2 = 0$ the matrix is 0, which is already diagonal. Hence the matrix is always diagonalizable.

Review

(i) Gram-Schmidt, QR-factorization (of a matrix with linearly independent columns, Gram-Schmidt applies to columns then scale)

(ii) Least-square problem: $\arg\min_x ||b - Ax||$. Normal equation: $A^TAx = A^Tb$. When the columns of A are linearly independent, A^TA is invertible and the solution is unique. Alternatively, let $A = QR$, then $R^TQ^TQRx = R^TQb$, hence $x = R^{-1}Qb$.

(iii) Symmetric matrix: A is symmetric iff $A^T = A$. Any symmetric matrix A can be written as P^TDP, where $P^TP = I$, D is diagonal.

(iv) Quadratic form in $\mathbb{R}^n$: $x \mapsto x^T Ax$. Change of variable: $A \mapsto P^T AP$, where P is the change-of-coordinate matrix.

Practice Problems

(i) Do QR decomposition for $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Answer: To find the column vectors of Q, do Gram-Schmidt on the column vectors of this matrix and then normalize their lengths to 1. The answer is:

$$Q = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{bmatrix} R = Q^T \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{\frac{1}{2}} & \sqrt{2} \\ 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{2}{3}} \\ 0 & 0 & \sqrt{\frac{1}{3}} \end{bmatrix}$$

(ii) Write down A, b such that the least square problem for $b - Ax$ does not have a unique solution.

Answer: The least square problem does not have a unique solution iff the columns of A are not linearly independent. Hence we can let them be e.g. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(iii) True or false: A and $P^T AP$ have the same eigenvalues.

Answer: False. A counterexample is $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(iv) True or false: A is a square matrix, λ is an eigenvalue of A, then λ^2 is an eigenvalue of $A^T A$.

Answer: False. A counterexample is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(v) Minimizes $x^T Ax$ for $Bx = b$.

Answer: There are many ways to state the solution.

If x^* minimizes $x^T Ax$ among all x such that $Bx = b$, then for any y such that $By = 0$, $(x^* + y)^T A(x^* + y) \geq (x^*)^T Ax^*$, hence $y^T Ax^* = 0$ and $y^T Ay \geq 0$. So the minimal can be found as follows:

- If $Bx = b$ is inconsistent, or if $P^T AP$ is not positive semidefinite, where P is the orthogonal projection to Nul B, there would be no minimal.
- If otherwise, let $x = x_0 + t_1 v_1 + \dots t_k v_k$ be the parametrized linear form of the solution of $Bx = b$. We can rewrite it as $x = x_0 + Vt$, where $V = [v_1 \quad v_2 \dots]$, $t = [t_1 \quad t_2 \dots]^T$. Then, solve for t^* in the equation $V^T Ax_0 + V^T AVt^* = 0$. Then the minimal of $x^T Ax$ with constrain $Bx = 0$ is $x_0^T A(x_0 + Vt^*)$.

Remark: You can also do it with Lagrange's multipliers.

May 11

Review

Singular value decomposition: $A = U\Sigma V^T$. Proof: $A^TA = VD^2V^T$, hence $(AV)^T(AV) = D^2$, i.e. the non-zero columns of AV form an orthogonal set with length equals the singular value, i.e. $AV = U\Sigma$ for some U, hence $A = U\Sigma V^T$.

Computation: find Σ and V by orthogonally diagonalizing A^TA, then let U be AV with column vectors filled and normalized.

Practice Problems

The final exam may not be as hard as these practice problems, so don't worry if you find these problems a bit challenging!

1. Find the SVD for $\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Answer: The answer is not unique. One answer is: $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, $U = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$.

2. Find the range of k such that $Q_k(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 - kx_1x_3$ is indefinite.

Answer: The matrix of Q_k is $\begin{bmatrix} 1 & 0 & -k/2 \\ 0 & 2 & 0 \\ -k/2 & 0 & 3 \end{bmatrix}$. So the characteristic polynomial is $(2-\lambda)((1-\lambda)(3-\lambda) - k^2/4) = (2-\lambda)(\lambda^2 - 4\lambda + 3 - k^2/4)$. Hence, it has both positive and negative eigenvalue iff $|k| \geq 2\sqrt{3}$.

3. True or false:

(i) If A is a symmetric matrix that defines an indefinite quadratic form, then 0 is a saddle point for discrete dynamical system $x_{k+1} = Ax_k$.

Answer: False. For example, $A = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$, then 0 is an attractor.

Remark: This is true if we ask about $x' = Ax$.

(ii) Any symmetric matrix of rank 1 can be written as $\pm xx^T$ for some $x \in \mathbb{R}^n$.

Answer: True. If A is symmetric, A can be orthogonally diagonalized, i.e. $A = PDP^T$ where P is orthogonal. Because the rank of A is 1, only one diagonal entries of D is non-zero. Suppose it is the k-th and its value is λ, then we can let x be the k-th column of P multiplies with $|\lambda|^{1/2}$.

(iii) Let A be any symmetric matrix, there are $a, b \in \mathbb{R}$ such that $\lim_{n\to\infty} b^{-n}(A+aI)^n$ exists and is a matrix of rank 1.

Answer: False. For example, A is the 2-by-2 identity matrix, then the limit is I if $b = a+1$, 0 if $|b| > |a+1|$, does not exist if otherwise, hence can never be of rank 1.

(iv) Let A be any matrix, λ be one of its (real) eigenvalues, then $|\lambda|$ is no larger than the largest singular value of A.

Answer: True. Let x be an eigenvector of the eigenvalue λ, then $|\lambda| = \sqrt{\frac{||\lambda x||^2}{||x||^2}} = \sqrt{\frac{||Ax||^2}{||x||^2}} = \sqrt{\frac{x^TA^TAx}{||x||^2}}$. Now let m be the largest singular value of A, then m^2 is the largest eigenvalue of A^TA. In other words, if $A^TA = UDU^{-1}$ is an orthogonal diagonalization of A^TA, the entries of D are all no larger than m^2. Hence $x^TA^TAx = (Ux)^TD(Ux) \le m^2||Ux||^2 = m^2||x||^2$. Hence $|\lambda| \le m$.

4. Let $\mathbb{P}_3$ be the vector space of polynomials of real coefficients of degree at most 3. $V = \{f \in \mathbb{P}^3 : \int_0^1 f(t)dt = 0\}$.

(i) Show that V is a subspace of $\mathbb{P}_3$.

Answer: $0 \in V$ because $\int_0^1 0dt = 0$. Addition and scaler multiplication are closed because integration is linear.

(ii) Find the dimension and a basis of V.

Answer: A polynomial $a_3x^3 + a_2x^2 + a_1x + a_0$ is in V iff $a_3/4 + a_2/3 + a_1/2 + a_0 = 0$. Hence, a basis can be chosen as $\{x^3 - 1/4, x^2 - 1/3, x - 1/2\}$. The dimension is 3. (The answer is not unique.)

(iii) Let $T : V \to V$ be a linear map defined as $T(f) = f' - f(1) + f(0)$. Write down the matrix for T under the basis you chose in (ii).

Answer: If you use the basis above, the matrix should be $\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$.

(iv) Find the rank, eigenvalues and eigenvectors of the matrix in (iii). Is it diagonalizable?

Answer: The rank is 2, the unique eigenvalue is 0 and its eigenspace is spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, or, the coefficient of $x - 1/2$. It does not span the whole vector space, hence the matrix is not diagonalizable.

5. Let M be the vector space of 2×2 symmetric matrices, find a basis of M and write down the matrix of quadratic form det. Is it positive definite, negative definite or indefinite?

Answer: A basis can be chosen as $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$. Denote the coefficient under this basis by (a, b, c), then $\det = ab - c^2$. The matrix is $\begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. It is indefinite because the determinant of a symmetric 2-by-2 matrix can be either positive or negative.