1. Introduction

Square-tiled translation surfaces are lattice surfaces because they are branched covers of the flat torus with a single branched point. Many non-square-tiled examples of lattice surfaces arise from "deforming" certain square-tiled surfaces. These includes Veech's regular n-gon, Ward's surface, and, more generally, the Bouw-Möller surfaces, which were discovered by Bouw-Möller [BM10]. They arises from abelian normal covers of the flat pillowcase. Hooper [Hoo13] gave a translation-surface-theoretic description of these examples by introducing a class of Thurston-Veech diagrams called "grid graphs", and Wright [Wri13]proved the equivalence of the construction in [Hoo13] and [BM10]. Here we formalize the construction of Bouw-Möller surfaces by studying the lattice surface that arises from deforming a certain marked square-tiled surface M according to a certain cohomology class $\alpha \in H^1(M, \Sigma; \mathbb{C})$. Our approach lead to a new class of Thurston-Veech diagram similar to Hooper's but not new lattice surfaces. We hope that it can provide some insights on what makes Bouw-Möller's construction works.

We will show that:

Theorem 1.1. *If a lattice surface arises from deforming an abelian branched cover of flat pillowcase M according to a relative cohomology class α, its affine diffeomorphism group is commensurable with the affine diffeomorphism group of M, and it also preserves the Thurston-Veech structure in the two diagonal directions, then its Thurston-Veech diagram must be one of the three types described in Section 4 and 5. As a consequence, the surface must be one of the followings cases:*

1. a branched cover of the regular $2n$-gon branched at the cone point.
2. a branched cover of a Bouw-Möller surface.
3. a branched cover of the regular n-gon branched at the mid-point of edges.

We start by describing the square-tiled surfaces we will deform. To do so, we use the same notation as in [Wu14], which we now review. As shown in Figure 1, let P be the flat pillowcase built from a pair of unit squares by identifying edges with the same label. The two squares are labeled as B^1 and B^2, the four cone points are labeled as z_1, z_2, z_3 and z_4, and the four edges are labeled as e_1, e_2, e_3, and e_4.

Let G be a finite abelian group. Let $\mathbf{g} = (g_1, g_2, g_3, g_4) \in G^4$ be a 4-tuple of elements in G such that $g_1 g_2 g_3 g_4 = 1$. We denote the simple loop around z_j that travels in counterclockwise direction on P, with base point in B^1, as l_j. The surface we will deal with, $M = M(G, \mathbf{g})$, is the branched abelian cover of the pillowcase P with deck group G induced by the group homomorphism $\pi_1(P - \{z_1, z_2, z_3, z_4\}) = \langle l_1, l_2, l_3, l_4 | l_1 l_2 l_3 l_4 = 1 \rangle \to G$ that sends the element in $\pi_1(P - \{z_1, z_2, z_3, z_4\})$ represented by l_j to $g_j \in G$.

The decomposition of P into two squares in figure 1 leads to a cell decomposition on $M(G, \mathbf{g})$. This cell structure can be described as $|G|$ copies of pairs of squares, labeled
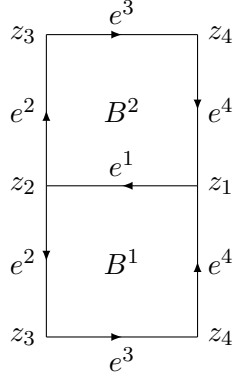
FIGURE 1.

by an element $g \in G$ and an index $k \in \{1, 2\}$ as B_g^k, as shown in figure 2, that are glued together by identifying edges with the same label in such a way that the arrows are in the same direction.
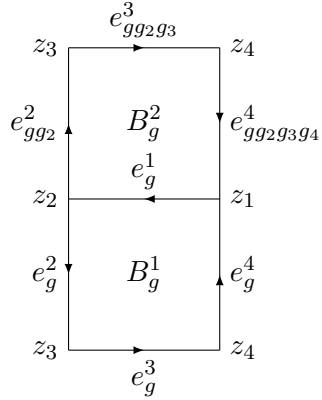

FIGURE 2.

For example, the cell structure of the surface $M(\mathbb{Z}/4, (1, 1, 1, 1))$ (the Wollmilchsau [FM08, HS]) is presented as in Figure 3.

Let Σ be the preimage of $\{z_1, z_2, z_3, z_4\}$. The flat structure on P induces a half translation structure on M with cone points in Σ. When the orders of g_1, g_2, g_3 and g_4 in G are all even, this induced half translation structure is also a translation structure, which is the only case we will deal with in this paper. If some of the g_i have odd orders, we can replace G by a larger group G' and replace M with a double cover M' such that the orders of all g_i in G' are even. As in [Wri13] and [Wu14], the action of G on the relative
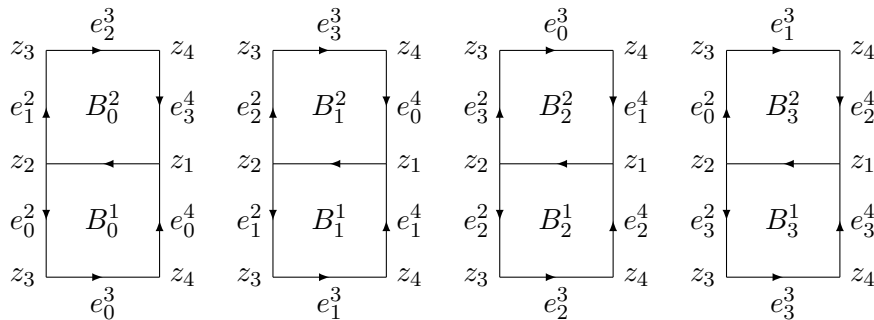
FIGURE 3.

cohomology of marked translation surface (M, Σ) induces a decomposition of $H^1(M, \Sigma; \mathbb{C})$ into irreducible representations. Hence we have decomposition $H^1(M, \Sigma; \mathbb{C}) = \bigoplus_\rho H^1(\rho)$, where ρ goes through all the irreducible representations of G, and $H^1(\rho)$ is the sum of all the irreducible G-subspaces isomorphic to ρ. This decomposition is preserved by a finite index subgroup of the Affine diffeomorphism group $\mathbf{Aff}(M)$, which we denote as Γ.

Given a translation surface (M, Σ) with a polygonal decomposition, and a 1-form $\alpha \in H^1(M, \Sigma; \mathbb{C})$, we now define the "deformation" of (M, Σ) according to α. We have:

Lemma 1.2. *Let (M, Σ) be a square-tiled translation surface. If a cohomology class $\alpha \in H^1(M, \Sigma; \mathbb{C})$ evaluated on the four sides of any square in M are the coordinates of the four sides of a convex quadrilateral with non-negative area, then there is a translation surface X with certain points identified, and a degree 1 map from M to X, such that the pull back of the translation structure of X is defined by α.*

Proof. We construct X and the map $M \to X$ as follows: every 2-cell B in M is mapped to a convex, possibly degenerate, quadrilateral on $\mathbb{C}$ with non-negative area, the coordinate of its four sides are given by α evaluated on the four sides of B. Each of the four sides of B is sent to the respective side of the quadrilateral linearly. The sides of those quadrilaterals are then glued to each other by isometry to form X. $\qquad \square$

Definition 1.1. We call X and the degree 1 map $i : M \to X$ in the previous lemma a *deformation* of M according to α.

For example, if we let $M = M(\mathbb{Z}/30, (1, 11, 29, 19))$, α be a 1-form corresponding to $(5, 3)$ Bouw-Möller surface (which we will describe in Section 5), then the deformation consists of four translation surfaces of the following shape glued together at their cone points as in Figure 4.

Labels $a \ldots h$ show the gluing of edges, and numbers show the image of 2-cells in M. Some of these images are convex quadrilaterals, some are triangles, and some (1,5,11,15 in
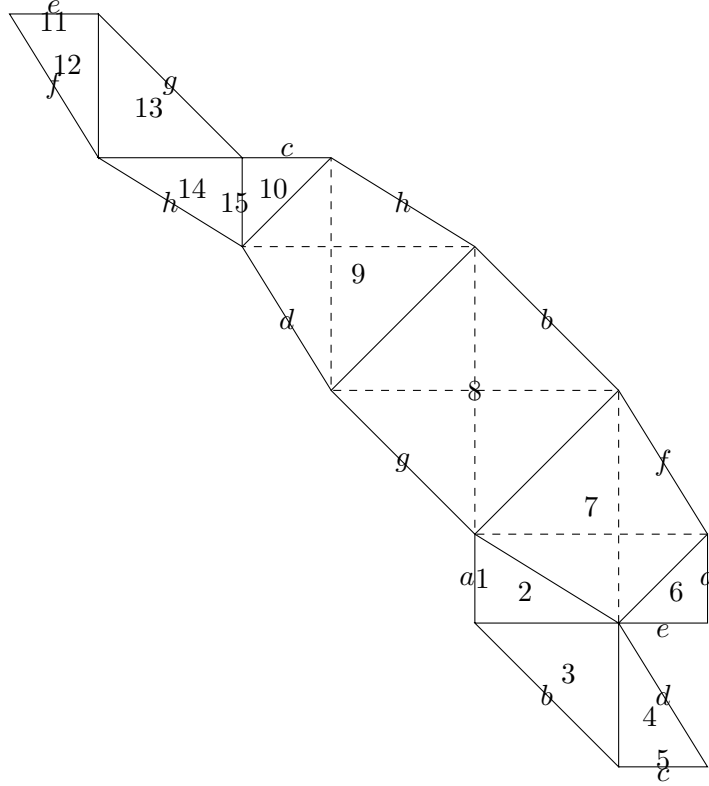
FIGURE 4.

the above figure) are collapsed into line segments.

The metric completion of $X - i(\Sigma)$ may have multiple connected components. Also, the deformation depends not only on α but also on the choice of cell structure on (M, Σ). We denote the deformation of M according to α as $M \to X(\alpha)$.

Definition 1.2. Let $i \colon (M, \Sigma) \to (X, \Sigma')$ be a deformation of (M, Σ). We say a homeomorphism $g \colon (M, \Sigma) \to (M, \Sigma)$ acts on X through an affine diffeomorphism $f \colon X \to X$, if the translation structure of X pulled back by $i \circ g$ and $f \circ i$ determine the same cohomology class in $H^1(M, \Sigma; \mathbb{C})$.

Remark 1.1. In the situation we will consider, we assume that a finite-index subgroup Γ' of the affine diffeomorphism group $\mathbf{Aut}(M)$ acts on lattice surface X through a finite index subgroup of $\mathbf{Aut}(X)$. Hence by Definition 1.2, Γ' preserves $N(\alpha) = span_{\mathbb{C}}(\alpha, \overline{\alpha})$, and the action is through a lattice in $U(1, 1)$.

The decomposition $H^1(M, \Sigma) = \bigoplus_\rho H^1(\rho)$ is Γ-invariant. By replacing Γ' with $\Gamma' \cap \Gamma$, we will, from now on, always assume that Γ' is a subgroup with finite index of Γ without loss of generality. As a consequence, the projection from $H^1(M, \Sigma)$ to each $H^1(\rho)$ is Γ'-equivariant. Hence we have, due to Schur's Lemma:

Proposition 1.3. *Under the assumption of Remark 1.1, the projection from $N(\alpha) = span_{\mathbb{C}}(\alpha, \overline{\alpha})$ to $H^1(\rho)$ is either 0, or a Γ'-isomorphism. In the latter case Γ' acts on $H^1(\rho)$ through a lattice in $U(1,1)$.*

$\square$

From this we know:

Corollary 1.1. *Under the assumption of Remark 1.1,*

$$N(\alpha) \subseteq \bigoplus_{H^1(\rho) \text{ is isomorphic to } N(\alpha) \text{ as } \Gamma'\text{-module}} H^1(\rho)$$

$\square$

In section 2 we review the concept of Thurston-Veech structure. In section 3 we review the discrete Fourier transform which is needed for the proof, and in section 4 and 5 we prove Theorem 1.1.

2. THURSTON-VEECH DIAGRAM

Recall that a Thurston-Veech structure [T+88] on a translation surface consists of two tuples of positive integers $\{r_i\}$ and $\{r'_j\}$, and a pair of transverse cylinder decompositions $\{C_i\}$ and $\{C'_j\}$, with moduli $\{M_i\}$ and $\{M'_j\}$ respectively, such that $M_{i_1}/M_{i_2} = r_{i_1}/r_{i_2}$, $M'_{j_1}/M'_{j_2} = r'_{j_1}/r'_{j_2}$. A Thurston-Veech structure on a translation surface induces two parabolic affine diffeomorphisms γ and γ' given by two multitwists on $\{C_i\}$ and $\{C'_j\}$ respectively. The intersection configuration of the two cylinder decompositions can be represented by a Thurston-Veech diagram, which is a ribbon graph constructed as follows: each vertex represents a cylinder, an edge between two vertices stands for an intersection of two cylinders, and the cyclic order among all edges associating with each vertex encodes the order of these intersections on the cylinder represented by this vertex.

On a square-tiled surface M, the two diagonal directions are periodic, and cylinder decompositions on these two directions form a Thurston-Veech structure. We call the cylinders from the bottom-left to top-right $\{C_i\}$, and the cylinders from the bottom-right to top-left $\{C'_j\}$. The Thurston-Veech diagram that arises from this Thurston-Veech structure can be constructed explicitly as follows: we get a square-tiled coned flat surface by gluing the squares with the same edge pairing as in M but with all the gluing directions flipped, which we denote as $\mathcal{F}M$. Now each vertex in the square tiling of $\mathcal{F}M$ corresponds to a cylinder in M formed by triangles, as illustrated in Figure 5.

Here the 4 triangles I, II, III and IV that form a cylinder in the bottom-left to top-right direction become the 4 triangles around P after the re-gluing, just as triangles that
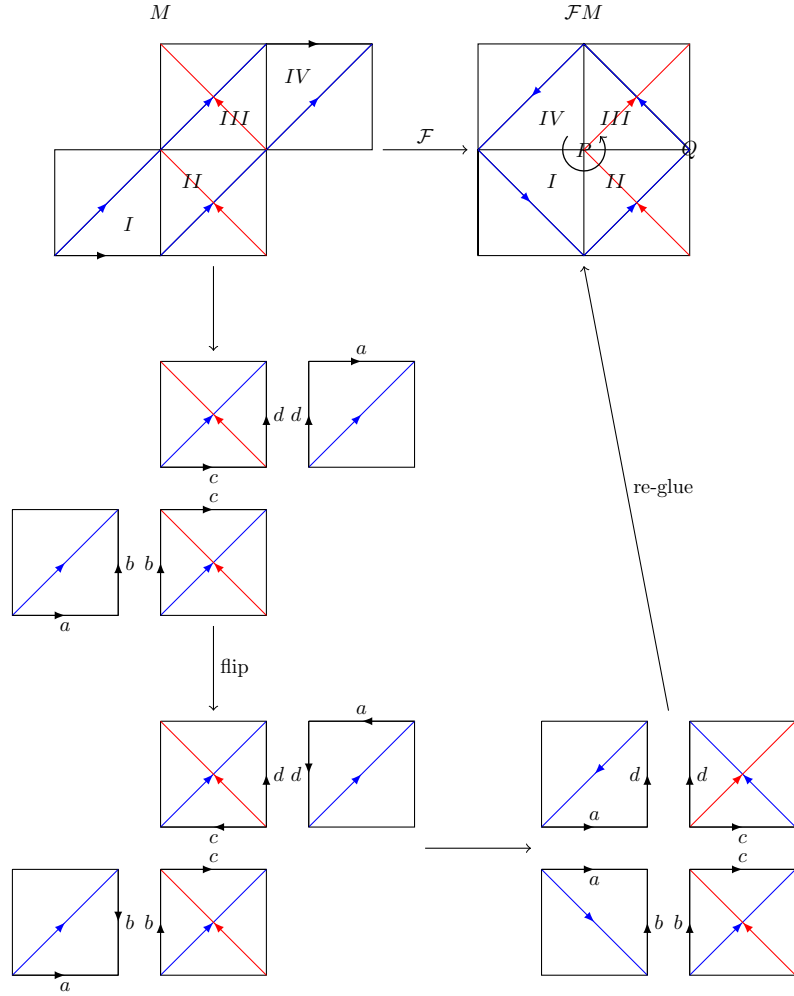
FIGURE 5.

form the cylinder between the two red arrows become the 4 triangles around Q after the re-gluing. Furthermore, the intersection of these two cylinders is represented by the edge PQ. The cyclic orders among the edges can be seen from blue and red arrows. Hence, the Thurston-Veech diagram is the 1-skeleton of $\mathcal{F}M$, with the cyclic orders in clockwise and counterclockwise direction alternatively by columns as illustrated in Figure 6.

In our case when M is a cover of the pillowcase, $\mathcal{F}M$ is always orientable. Because in the gluing process described above, any two adjacent squares are always glued with orientation reversed, we can define an orientation on $\mathcal{F}M$ as either the orientation of all the B_g^1 after the regluing, or the opposite of the orientation of all B_g^2 after regluing. Also, because M is
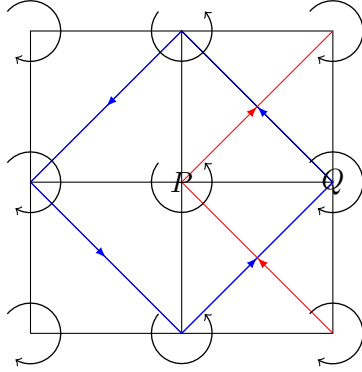
FIGURE 6.

a normal cover, all cylinders in the same diagonal direction have the same circumference and width hence the same moduli, i.e. $r_i = r'_j = 1$ for all i, j.

Definition 2.1. We say that a 1-form $\alpha \in H^1(M, \Sigma)$ *preserves* a cylinder decomposition $\{C_i\}$ on M, if α evaluated on all paths contained in all the boundary curves of these cylinders are parallel. We say α *preserves* a Thurston-Veech structure, if it preserves both cylinder decompositions, and the multitwists γ, γ', which are induced by to the Thurston-Veech structure, preserve $span_{\mathbb{C}}(\alpha, \overline{\alpha})$.

A cohomology class α that satisfies the assumption of Remark 1.1 must preserve at least two parabolic elements, and, thereby, two cylinder decompositions. Here, we assume that it preserves the two cylinder in the two diagonal directions of M. We denote the set of elements of $H^1(M, \Sigma; \mathbb{C})$ that preserves the two cylinder decomposition in the diagonal directions as $\mathcal{L}(M, \Sigma)$, and the set of those elements in $\mathcal{L}$ that further preserve the Thurston-Veech structure in the diagonal directions as $\mathcal{N}(M, \Sigma)$. Furthermore, we denote by $\mathcal{L}'$ and $\mathcal{N}'$ the set of those elements in $\mathcal{L}$ and $\mathcal{N}$ whose value on bottom-left-to-top-right diagonals are in $\mathbb{R}$, and whose value on bottom-right-to-top-left diagonals are in $\sqrt{-1}\mathbb{R}$. Any element in $\mathcal{L}$ or $\mathcal{N}$ can be made into an element in $\mathcal{L}'$ or $\mathcal{N}'$ after an affine transformation.

The circumference, width and moduli of C_i under α is defined as the base, height and moduli of the parallelogram formed by α evaluated on the boundary circle and α evaluated on a path that crosses the cylinder once from right to left.

Let $V = V(\mathcal{F}M)$ be the space of real valued functions on the vertices of $\mathcal{F}M$. Let $\Psi : \mathcal{L}' \to V$ be the map defined as follows: given any element $\alpha \in \mathcal{L}'$, $\Psi(\alpha)$ send a vertex to the width under α of the cylinder it represents. It is a bijective map. Because $r_i = r'_j = 1$, the fact that multitwists γ and γ' preserve $span_{\mathbb{C}}(\alpha, \overline{\alpha})$ implies that any C_i with non-zero width have the same moduli, and those with width 0 also have 0 circumference. After an

affine action we can make all the cylinders to have either identical moduli $\lambda > 0$ or zero width and zero circumference. Hence, by [T$^+$88], we have:

Lemma 2.1. [T$^+$88] *Let A be the adjacency matrix of the Thurston-Veech diagram, then up to an affine action, α satisfies $A(\Psi(\alpha)) = \lambda\Psi(\alpha)$.*

$\square$

When $\alpha \in \mathcal{L}'$, the assumption of Lemma 1.1 can be further rewritten as a condition on the image of Ψ:

Lemma 2.2. *Assuming $\alpha \in \mathcal{L}'$, the assumption of Lemma 1.1 is equivalent to the fact that no pair of adjacent vertices of $\mathcal{F}M$ can be assigned values of opposite signs by $\Psi(\alpha)$, and there is an edge such that neither of its end points are assigned 0.*

Proof. If assumption of Lemma 1.1 holds, any square B_j^k will be sent to a convex quadrilateral with non-negative area under the flat structure defined by α. The two diagonals of this quadrilateral cut it into 4 small triangles of non-negative area. If two non-degenerate cylinders in the cylinder decompositions described above intersect, its intersection must be a rectangle formed by two of those small triangles hence must have non-negative signed area, i.e. $\Psi(\alpha)$ can not assign values of opposite signs for any pair of adjacent vertices. At least one of these rectangles must have positive area, hence there is at least one pair of adjacent vertices such that $\Psi(\alpha)$ is non-zero on both.

On the other hand, if $\Psi(\alpha)$ does not assign values of opposite signs for any pair of adjacent vertices, we can build a translation surface X as follows: each parallelogram formed by the intersection of some C_i and C_j' are sent to a $\Psi(\alpha)(v_i)$-by-$\Psi(\alpha)(v_j)$ rectangle, where v_i and v_j are vertices in $\mathcal{F}M$ that represent C_i and C_j' respectively. As illustrated in Figure 7, the sides of these rectangles are glued in the same order as the gluing of the parallelograms on M, which means that α satisfies the assumption of Lemma 1.1.
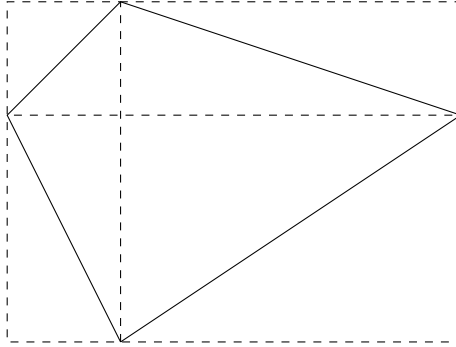

FIGURE 7.

$\square$

Remark 2.1. When α satisfies the assumption of Lemma 1.1, we can decompose $X(\alpha)$ into a union of (possibly degenerated) cylinders in two ways, and choose the map $i : M \to X(\alpha)$ such that these two cylinder decompositions are the images of $\{C_i\}$ and $\{C'_j\}$.

Let $\mathcal{S}$ be the category of square-tiled surfaces with morphisms being continuos maps that send squares isometrically to squares. Our construction of $\mathcal{F}M$, $\mathcal{L}(M)$, $\mathcal{N}(M)$ and $V(\mathcal{F}M)$ from M are functorial. More precisely,

Lemma 2.3. *A $\mathcal{S}$-morphism $\xi : M \to M'$ induces canonically a $\mathcal{S}$-morphism $\mathcal{F}\xi : \mathcal{F}M \to \mathcal{F}M'$, $\xi^* : \mathcal{L}'(M') \to \mathcal{L}'(M)$, and $(\mathcal{F}\xi)^* : V(\mathcal{F}M') \to V(\mathcal{F}M)$ and the following diagram commute:*

$$\begin{array}{ccc} \mathcal{L}'(M') & \to & \mathcal{L}'(M) \\ \downarrow \Psi & & \downarrow \Psi \\ V(\mathcal{F}M') & \to & V(\mathcal{F}M) \end{array}$$

Furthermore, ξ^* sends $\mathcal{N}'$ to $\mathcal{N}'$.

$\square$

Remark 2.2. $(\mathcal{F}\xi)^*$ can be described more concretely as follows: for any $f \in V(\mathcal{F}M')$, any vertex in $\mathcal{F}M$, $(\mathcal{F}\xi)^* f(v) = f(\mathcal{F}\xi(v))$ if $\mathcal{F}\xi$ preserves the cyclic order among the edges associated to v, and $(\mathcal{F}\xi)^* f(v) = -f(\mathcal{F}\xi(v))$ if it reverses the cyclic order.

Remark 2.3. $\mathcal{F}^2$ is the same as the identity functor $1_{\mathcal{S}}$. Hence $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an isomorphism.

Now we consider the question of finding $\alpha \in \mathcal{N}'$ that satisfies the assumption of Lemma 1.1 and that a subgroup of $\mathbf{Aff}(M)$ of finite index acts on $N(\alpha)$ as a lattice. We say such α satisfies property (L). Now we analyze all possible 1-forms with property (L).

Let e be the identity element in G. Let γ_1 be the element in Γ that preserve the button-right corner of B_e^1 and has derivative $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$; let γ_2 be the element in Γ that preserve the button-right corner of B_e^1 and has derivative $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$; let γ_3 be the element in Γ that preserve the button-right corner of B_e^1 and has derivative $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$, and let γ_4 be the element in Γ that preserve the button-right corner of B_e^1 and has derivative $\begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$. These elements are liftings of the Dehn twists in the horizontal, vertical and the two diagonal directions on P.

Let $\alpha \in \mathcal{N}'$. By Definition 2.1, some power of γ_3 and γ_4 acts on $X(\alpha)$ as parabolic affine automorphisms, hence their action on $N(\alpha)$ are parabolic. Because a power of γ_3 acts on $H^1(\rho)$ as parabolic map if and only if $\rho(g_1 g_3) = 1$, by Corollary 1.1, $N(\alpha) \subset \oplus_{\rho \in \{\rho : \rho(g_1)\rho(g_3) = 1\}} H^1(\rho)$. Hence, α is the pull back of some $\alpha' \in \mathcal{N}'(M(G/(g_1 g_3), \mathbf{g}))$ by a $ord_G(g_1 g_3)$-fold branched cover $M \to M_1 = M(G/(g_1 g_3), \mathbf{g})$. Hence $X(\alpha)$ is a branched

cover of $X(\alpha')$ branching at $i(\Sigma) \subset \Sigma'$.

Because the deck group of M_1 is $G_1 = G/(g_1 g_3)$, $g_1 g_3 = g_2 g_4 = e$ in G_1. As a consequence, $\mathcal{F}M_1$ is a compact orientable flat surface with no cone points, hence it must be a square-tiled torus $\mathbb{R}^2/L$ where $L \subset \mathbb{Z}^2$. Furthermore, from the construction of $\mathcal{F}M$ we know that each horizontal cylinder in M corresponds to a horizontal cylinder of the same width and circumference in $\mathcal{F}M$. Hence $(2ord_G(g_1 g_2), 0) \in L$. Similarly, $(0, 2ord_G(g_1 g_4)) \in L$. Let M_2 be a (not necessarily connected) square tiled surface such that $\mathcal{F}M_2$ is a $2ord_G(g_1 g_2)$-by-$2ord_G(g_1 g_4)$ torus, then M_2 is a branched cover of M_1. By the construction of $\mathcal{F}$, M_2 is also a branched abelian cover. Furthermore, the pull back of any $\alpha' \in \mathcal{N}(M, \Sigma)$ satisfying property (L) would also satisfy property (L). Hence, we have:

Proposition 2.4. *Any α satisfying property (L) is related to an $\alpha_0 \in \mathcal{N}'(M_2)$ satisfying property (L) by finite branched covers, where $\mathcal{F}M_2$ is a $2ord_G(g_1 g_2)$-by-$2ord_G(g_1 g_4)$ torus $\mathbb{R}^2/2ord_G(g_1 g_2)\mathbb{Z} \times 2ord_G(g_1 g_4)\mathbb{Z}$*

$\square$

Hence, from now on, we always let $\mathcal{F}M$ be a $2s$-by-$2t$ rectangular torus.

3. Discrete Fourier Transform

To apply Corollary 1.1, we need to find a decomposition of V that is compatible with the decomposition $H^1(M, \Sigma; \mathbb{C}) = \bigoplus_\rho H^1(\rho)$. We can obtain such a decomposition by Discrete Fourier Transform.

Let $V(\mathcal{F}M) = V(T_{s,t})$ be the space of real-valued functions on $\mathbb{Z}^2/(2s\mathbb{Z} \times 2t\mathbb{Z})$. By discrete Fourier transform, V has the following direct-sum decomposition: $V_{s,t} = \bigoplus_{\lambda,\mu} V_{s,t}^{\lambda,\mu}$, where λ, μ are integers, such that $0 \le \lambda < s/2$, $0 \le \mu < t$ if s is even, $0 \le \lambda < s/2$, $0 \le \mu < t$, or $\lambda = (s-1)/2$ and $\mu < t/2$ if s is odd, and $V_{s,t}^{\lambda,\mu} = span_\mathbb{R}(f_{0,0}, f_{0,1}, f_{1,0}, f_{1,1}, g_{0,0}, g_{0,1}, g_{1,0}, g_{1,1})$, where:

$$f_{0,0}(x,y) = \begin{cases} \sin(2x\pi\lambda/2s)\sin(2y\pi\mu/2t), & \text{if } x+y \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

$$f_{0,1}(x,y) = \begin{cases} \sin(2x\pi\lambda/2s)\cos(2y\pi\mu/2t), & \text{if } x+y \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

$$f_{1,0}(x,y) = \begin{cases} \cos(2x\pi\lambda/2s)\sin(2y\pi\mu/2t), & \text{if } x+y \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

$$f_{1,1}(x,y) = \begin{cases} \cos(2x\pi\lambda/2s)\cos(2y\pi\mu/2t), & \text{if } x+y \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

$$g_{0,0}(x,y) = \begin{cases} \sin(2x\pi\lambda/2s)\sin(2y\pi\mu/2t), & \text{if } x+y \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

$$g_{0,1}(x,y) = \begin{cases} \sin(2x\pi\lambda/2s)\cos(2y\pi\mu/2t), & \text{if } x+y \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

$$g_{1,0}(x,y) = \begin{cases} \cos(2x\pi\lambda/2s)\sin(2y\pi\mu/2t), & \text{if } x+y \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

$$g_{1,1}(x,y) = \begin{cases} \cos(2x\pi\lambda/2s)\cos(2y\pi\mu/2t), & \text{if } x+y \text{ is odd} \\ 0. & \text{otherwise} \end{cases}$$

This decomposition is related to, but not the same as the eigenspace decomposition of discrete Laplacian.

By the functorial property, the G-action on M induces an action on $\mathcal{F}M$ which induces actions on various $V^{\lambda,\mu}$. More precisely, the action of g_i on $\mathcal{F}M$, $i = 1, 2, 3, 4$, are translations in the four diagonal directions. By calculation, we know that $\Psi^{-1}(V^{\lambda,\mu}) \subset H^1(\rho_{++}^{\lambda,\mu}) + H^1(\rho_{-+}^{\lambda,\mu}) + H^1(\rho_{+-}^{\lambda,\mu}) + H^1(\rho_{--}^{\lambda,\mu})$, where:

$$\begin{array}{ll}
\rho_{++}^{\lambda,\mu}(g_1) = e^{i\pi(1+\lambda/s+\nu/t)} & \rho_{++}^{\lambda,\mu}(g_2) = e^{i\pi(1-\lambda/s+\nu/t)} \\
\rho_{++}^{\lambda,\mu}(g_3) = e^{i\pi(1-\lambda/s-\nu/t)} & \rho_{++}^{\lambda,\mu}(g_4) = e^{i\pi(1+\lambda/s-\nu/t)} \\
\rho_{-+}^{\lambda,\mu}(g_1) = e^{i\pi(1-\lambda/s+\nu/t)} & \rho_{-+}^{\lambda,\mu}(g_2) = e^{i\pi(1+\lambda/s+\nu/t)} \\
\rho_{-+}^{\lambda,\mu}(g_3) = e^{i\pi(1+\lambda/s-\nu/t)} & \rho_{-+}^{\lambda,\mu}(g_4) = e^{i\pi(1-\lambda/s-\nu/t)} \\
\rho_{+-}^{\lambda,\mu}(g_1) = e^{i\pi(1+\lambda/s-\nu/t)} & \rho_{+-}^{\lambda,\mu}(g_2) = e^{i\pi(1-\lambda/s-\nu/t)} \\
\rho_{+-}^{\lambda,\mu}(g_3) = e^{i\pi(1-\lambda/s+\nu/t)} & \rho_{+-}^{\lambda,\mu}(g_4) = e^{i\pi(1+\lambda/s+\nu/t)} \\
\rho_{--}^{\lambda,\mu}(g_1) = e^{i\pi(1-\lambda/s-\nu/t)} & \rho_{--}^{\lambda,\mu}(g_2) = e^{i\pi(1+\lambda/s-\nu/t)} \\
\rho_{--}^{\lambda,\mu}(g_3) = e^{i\pi(1+\lambda/s+\nu/t)} & \rho_{--}^{\lambda,\mu}(g_4) = e^{i\pi(1-\lambda/s+\nu/t)}
\end{array}$$

From [BM10], [Wri13] or Section 7 of [Wu14], we know that if the action of Γ on $\mathbb{P}H^1(\rho)$ for any of the $\rho_{\pm\pm}$ above is commensurable to a hyperbolic triangle group with parabolic element, then either (a) $\lambda = 0$ and $\nu|t$, or $\nu = 0$ and $\lambda|s$ or (b) $\lambda|s$ and $\nu|t$. In case (a) it is commensurable to the (∞, ∞, n)-triangle group, while in case (b) it is commensurable to a (∞, m, n)-triangle group. By Corollary 1.1, if α satisfies property (L) then $\Psi(\alpha)$ must be in one of these two types of $V^{\lambda,\mu}$.

4. THE (∞, ∞, n) CASE

In this case, without loss of generality we can assume $\lambda = s/n$ and $\nu = 0$. Figure 8 shows the sign of the function $f_0 = f_{0,0} + g_{0,0}$ for $n = 5$ on vertices of $\mathcal{F}M$.

Points on which $f_0 = f_{0,0} + g_{0,0}$ is positive or negative form vertical stripes, which are separated by columns of zeros, hence $\Psi^{-1}(f_0)$ defines a singular translation structure on M by Lemma 2.2. By computation, this translation structure is tiled by regular $2n$-gons.
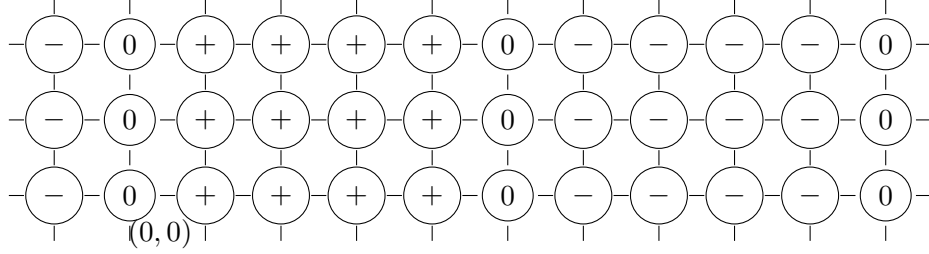
FIGURE 8.

These lattice surfaces were first discovered by Veech.

Now we have:

Proposition 4.1. *If $\alpha \in \mathcal{N}'$ that satisfies condition (L) such that $\Psi(\alpha) \in V^{s/n,0}$, then up to an affine action and an automorphism of the square-tiled surface M, $\Psi(\alpha) = f_0$. Hence, $X(\alpha)$ is tiled by regular $2n$-gons up to a $SL(2, \mathbb{R})$-action.*

Proof. Firstly, because the sum of the values of $\Psi(\alpha)$ is 0, by Lemma 2.2, the set of vertices on which $\Psi(\alpha)$ is positive and the set of vertices on which $\Psi(\alpha)$ is negative have to be separated by vertices on which $\Psi(\alpha)$ is 0. Hence, $\Psi(\alpha)$ must reaches 0 at a certain vertex. After a G action as well as relabeling if necessary, we can assume this vertex to be $(0,0)$ without loss of generality. Hence $\Psi(\alpha)(-1,1)\Psi(\alpha)(1,1) < 0$ or $\Psi(\alpha)(x,y) = 0$ for all $x + y$ even. If $\Psi(\alpha)(-1,1)\Psi(\alpha)(1,1) < 0$, by Lemma 2.2, $\Psi(\alpha)(0,1) = 0$, hence $\Psi(\alpha)(x,y) \in f_{0,0}\mathbb{R} + g_{0,0}\mathbb{R}$, hence α is affine equivalent to α_0.

The other case, $\Psi(\alpha)(x,y) = 0$ for all $x + y$ even, contradicts with our assumption according to Lemma 2.2. $\qquad\square$

5. The (∞, m, n) case, Bouw-Möller surfaces

Now we deal with the (∞, m, n) case. Let $f_1 = f_{0,0} + g_{0,0}$. Points on which f_1 is positive or negative form rectangles, which are separated by columns and rows of zeros, hence $\Psi^{-1}(f_1)$ defines a singular translation structure on M by Lemma 2.2. Figure 9 shows the sign of f_1 when $m = 5, n = 3$.

By deleting vertices that are assigned 0 width in the 1-skeleton on $T_{s,t}$ we can get the grid graph described in [Hoo13], hence due to Lemma 2.1 the metric completion of each component of $(X(\Psi^{-1}(f_1)) - \Sigma')_*$ is the (m, n) Bouw-Möller surface.

For example, if we label the squares of $\mathcal{F}M$ in the Figure 9 above as in Figure 10, then one component of $X(\Psi^{-1}(f_1))$ becomes Figure 11, and $X(\Psi^{-1}(f_1))$ is a union of copies of such translation surfaces glued together at the cone point.
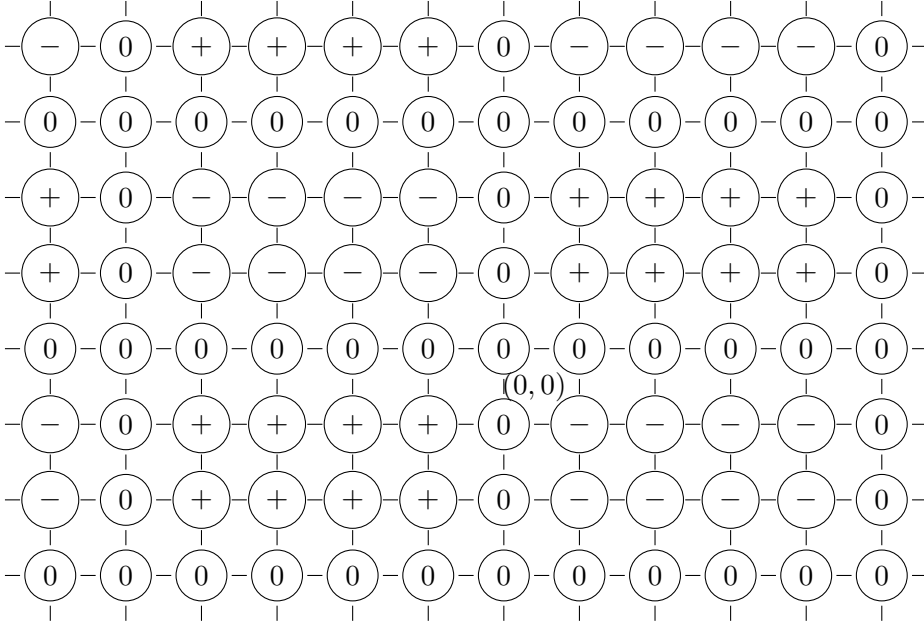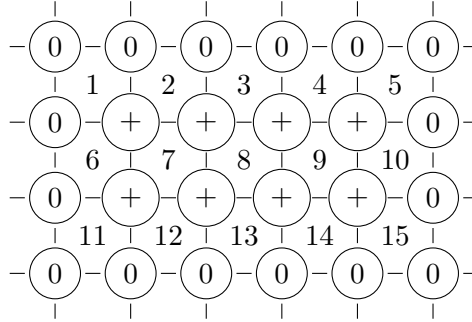
Figure 9.

Figure 10.

Also, when $m = n$, $f_2 = f_{0,0} + f_{1,1} + g_{0,0} + g_{1,1}$ and $f_3 = f_{0,1} + f_{0,1} + g_{1,0} + g_{1,0}$ also define lattice surfaces. They are affine equivalent to branched covers of the regular n-gon branched at the midpoints of its sides. Figure 12 shows the sign of f_2 for $m = n = 5$.

Similar to the previous section, we have:

Proposition 5.1. *If $\alpha \in \mathcal{N}'$ that satisfies condition (L) such that $\Psi(\alpha) \in V^{s/n, t/m}$, then up to affine action and the automorphism of M, $\Psi(\alpha)$ is either f_1 or f_2.*

Proof. Due to Lemma 2.1, we can further assume that $A\Psi(\alpha) = \lambda\Psi(\alpha)$ for some constant λ. From this and the Peron-Frobenious theorem we know that the sign of $\Psi(\alpha)$ completely
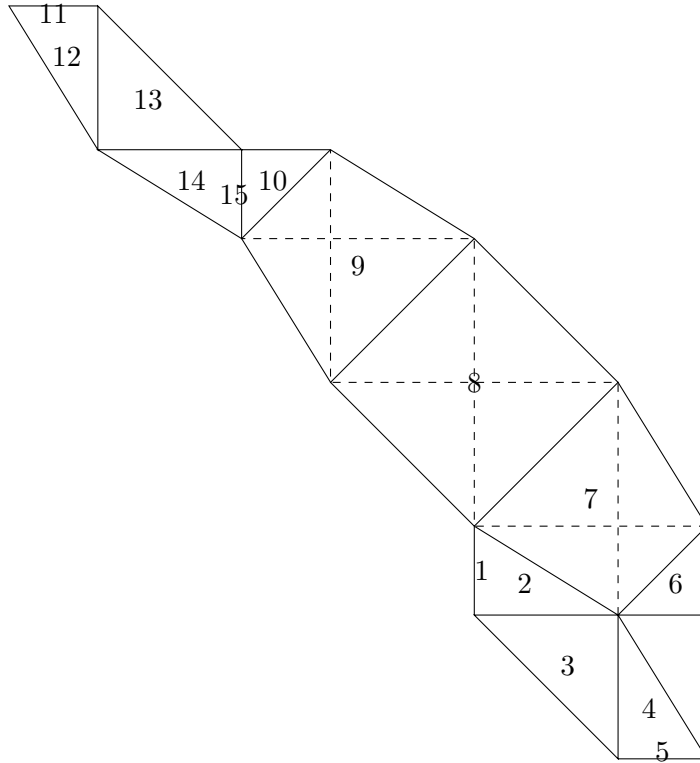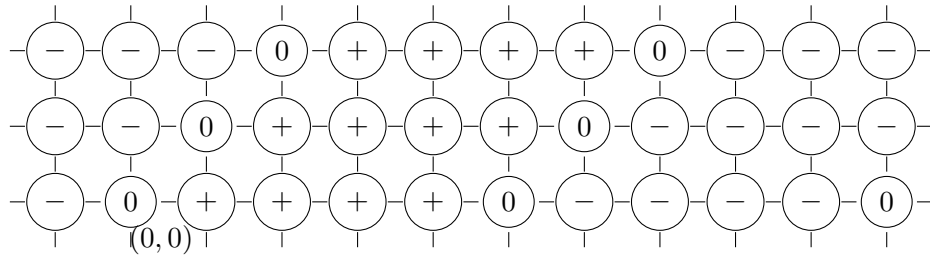
FIGURE 11.

FIGURE 12.

determines $\Psi(\alpha)$ up to scaling.

By the formula for $f_{i,j}$ and $g_{i,j}$, $i,j = 0,1$ in Section 3 and the fact that $A\Psi(\alpha) = \lambda\Psi(\alpha)$, $\Psi(\alpha) = C_1 \sin(\pi x/n + A)\cos(\pi y/m) + C_2 \sin(\pi x/n + B)\sin(\pi y/m)$ for some constants C_1, C_2, A and B, hence it restricted to any row is a function of the form $\psi(x) = C\sin(\pi x/n + a)$ for some constants C and a, and it is a function of the form $\psi'(x) = C'\sin(\pi y/m + a')$ for some constants C' and a' when restricted to any column. Furthermore, by Lemma 2.2, the

set of vertices on which $\Psi(\alpha)$ is positive and the set of vertices on which $\Psi(\alpha)$ is negative have to be separated by vertices on which $\Psi(\alpha)$ is 0, hence $\Psi(\alpha)$ restricted to any row of vertices must be one of the following cases:

(a) 0.

(b) $C\sin(\pi(x-k)/n)$, $k \in \mathbb{Z}$. It reaches 0 $\frac{2s}{n}$ times.

Similarly, $\Psi(\alpha)$ restricted to any column of vertices must be one of the following cases:

(a) 0.

(b) $C'\sin(\pi(x-k')/m)$, $k' \in \mathbb{Z}$. It reaches 0 $\frac{2t}{m}$ times.

Because of Lemma 2.2, at least one row or column will be in case (b). Furthermore, we have:

Lemma 5.2. *If a row or a column of type (a) is next to a row or a column of type (b), the subgraph spanned of vertices on which $\Psi(\alpha)$ is non-zero is a union of grid graphs as defined in* [Hoo13], *hence by Lemma 2.1 it must be a cover of a Bouw-Möller surface.*

Proof. If a row of type (a) is next to a row of type (b), without loss of generality we let the row of type (a) be the 0-th row and the row od type (b) be the 1st. Because $\Psi(\alpha) = C_1 \sin(\pi x/n + A)\cos(\pi y/m) + C_2 \sin(\pi x/n + B)\sin(\pi y/m)$ and $\Psi(\alpha)(x,y) = 0$, $C_1 = 0$. Furthermore, because the 1st row is of type (b), $nB/\pi \in \mathbb{Z}$. Hence the subgraph spanned of vertices on which $\Psi(\alpha)$ is non-zero is a union of grid graphs as defined in [Hoo13]. The argument for columns is the same. □

Now we only need to deal with the case when $\Psi(\alpha)$ restricted to any row or column of vertices are all of type (b). Because all columns are of type (b) there must be $\frac{4st}{m}$ zeros, and because all rows are of type (b) there must be $\frac{4st}{n}$ zeros. Hence, $m = n$. The only way for these zero vertices to separate the other vertices of $T_{s,t}$ into positive and negative parts is by aligning in the diagonal direction. In other words, after scaling and relabeling, $\Psi(\alpha)$ is $f_{0,0} + f_{1,1} + g_{0,0} + g_{1,1}$ or $f_{0,1} + f_{0,1} + g_{1,0} + g_{1,0}$. □

Proof of Theorem 1.1. Theorem 1.1 is a corollary of Proposition 4.1 and Proposition 5.1. □

References

[BM10] I. I. Bouw and M. Möller. Teichmüller curves, triangle groups, and Lyapunov exponents. *Ann. of Math.*, 172(2):139–185, 2010.

[FM08] G. Forni and C. Matheus. An example of a Teichmüller disk in genus 4 with degenerate Kontsevich-Zorich spectrum. 2008. arXiv preprint arXiv:0810.0023.

[Hoo13] W. P. Hooper. Grid graphs and lattice surfaces. *International Mathematics Research Notices*, 12, 2013.

[HS] F. Herrlich and G. Schmithüsen. An extraordinary origami curve. *Math. Nachr.*, (2):219–237.

[T+88] William P Thurston et al. On the geometry and dynamics of diffeomorphisms of surfaces. *Bulletin (new series) of the American Mathematical Society*, 19(2):417–431, 1988.

[Wri13] A. Wright. Schwarz triangle mappings and Teichmüller curves: the Veech-Ward-Bouw-Möller curves. *Geometric and Functional Analysis*, 23(2):776–809, 2013.

[Wu14] C. Wu. Affine diffeomorphisms on the relative cohomology of abelian covers of the flat pillowcase. 2014.